

# CS477 Formal Software Dev Methods

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# Propositional Logic

## The Language of Propositional Logic

- Begins with constants  $\{\mathbf{T}, \mathbf{F}\}$
- Assumes countable set  $AP$  of **propositional variables**, a.k.a. **propositional atoms**, a.k.a. **atomic propositions**
- Assumes **logical connectives**:  $\wedge$  (and);  $\vee$  (or);  $\neg$  (not);  $\Rightarrow$  (implies);  $\Leftrightarrow$  = (if and only if)
- The set of **propositional formulae**  $PROP$  is the inductive closure of these as follows:
  - $\{\mathbf{T}, \mathbf{F}\} \subseteq PROP$
  - $AP \subseteq PROP$
  - if  $A \in PROP$  then  $(A) \in PROP$  and  $\neg A \in PROP$
  - if  $A \in PROP$  and  $B \in PROP$  then  $(A \wedge B) \in PROP$ ,  $(A \vee B) \in PROP$ ,  $(A \Rightarrow B) \in PROP$ ,  $(A \Leftrightarrow B) \in PROP$ .
  - Nothing else is in  $PROP$
- Informal definition; formal definition requires math foundations, set theory, fixed point theorem ...

# Semantics of Propositional Logic: Model Theory

Model for Propositional Logic has three parts

- Mathematical set of **values** used as meaning of propositions
- Interpretation function giving meaning to props built from logical connectives, via structural recursion

Standard Model of Propositional Logic

- $\mathcal{B} = \{\text{true}, \text{false}\}$  boolean values
- $v : AP \rightarrow \mathcal{B}$  a **valuation**
- Interpretation function . . .

# Semantics of Propositional Logic: Model Theory

## Standard Model of Propositional Logic (cont)

- Standard interpretation  $\mathcal{I}_v$  defined by structural induction on formulae:
  - $\mathcal{I}_v(\mathbf{T}) = \text{true}$  and  $\mathcal{I}_v(\mathbf{F}) = \text{false}$
  - If  $a \in AP$  then  $\mathcal{I}_v(a) = v(a)$
  - For  $p \in PROP$ , if  $\mathcal{I}_v(p) = \text{true}$  then  $\mathcal{I}_v(\neg p) = \text{false}$ , and if  $\mathcal{I}_v(p) = \text{false}$  then  $\mathcal{I}_v(\neg p) = \text{true}$
  - For  $p, q \in PROP$ 
    - If  $\mathcal{I}_v(p) = \text{true}$  and  $\mathcal{I}_v(q) = \text{true}$ , then  $\mathcal{I}_v(p \wedge q) = \text{true}$ , else  $\mathcal{I}_v(p \wedge q) = \text{false}$
    - If  $\mathcal{I}_v(p) = \text{true}$  or  $\mathcal{I}_v(q) = \text{true}$ , then  $\mathcal{I}_v(p \vee q) = \text{true}$ , else  $\mathcal{I}_v(p \vee q) = \text{false}$
    - If  $\mathcal{I}_v(q) = \text{true}$  or  $\mathcal{I}_v(p) = \text{false}$ , then  $\mathcal{I}_v(p \Rightarrow q) = \text{true}$ , else  $\mathcal{I}_v(p \Rightarrow q) = \text{false}$
    - If  $\mathcal{I}_v(p) = \mathcal{I}_v(q)$  then  $\mathcal{I}_v(p \Leftrightarrow q) = \text{true}$ , else  $\mathcal{I}_v(p \Leftrightarrow q) = \text{false}$

# Truth Tables

Interpretation function often described by **truth table**

$p$	$q$	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
true	true					
true	false					
false	true					
false	false					

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true	false	false				
false	true	true				
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false	true	true	false	true		
false	false	true	false	false		



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true	false	false	false	true	false	
false	true	true	false	true	true	
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true	false	false	false	true	false	false
false	true	true	false	true	true	false
false	false	true	false	false	true	true

# Modeling Propositional Formulae

- $(\mathcal{B}, \mathcal{I})$  is the **standard model** of proposition logic
- Given valuation  $v$  and proposition  $p \in PROP$ , write  $v \models p$  iff  $\mathcal{I}_v(p) = \text{true}$ 
  - More fully written as  $\mathcal{B}, \mathcal{I}, v \models p$
  - Say  $v$  **satisfies**  $p$ , or  $v$  **models**  $p$
  - Write  $v \not\models p$  if  $\mathcal{I}_v(p) = \text{false}$
- $p$  is **satisfiable** if there exists valuation  $v$  such that  $v \models p$
- $p$  is **valid**, a.k.a. a **tautology** if for every valuation  $v$  we have  $v \models p$
- $p$  is **logically equivalent** to  $q$ ,  $p \equiv q$  if for every valuation,  $v$ , we have  $v \models p$  iff  $v \models q$ 
  - Claim: Logical equivalence is an equivalence relation

# Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

$A$	$B$	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true			
true	false			
false	true			
false	false			

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$A$	$B$	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true	true		
true	false	false		
false	true	true		
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true	false	false	true	
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$A$	$B$	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true	true	true	true
true	false	false	true	true
false	true	true	true	true
false	false	true	false	true

# Example Tautology – Your Turn



# Example: Logical Equivalence

$$A \Rightarrow B \equiv ((\neg A) \vee B)$$

$A$	$B$	$A \Rightarrow B$	$\neg A$	$(\neg A) \vee B$
true	true	true	false	true
true	false	false	false	false
false	true	true	true	true
false	false	true	true	true

# More Useful Logical Equivalences

$$\begin{array}{lll} \neg\neg A \equiv A & \neg\mathbf{T} \equiv \mathbf{F} & \neg\mathbf{F} \equiv \mathbf{T} \\ (A \vee A) \equiv A & (A \vee B) \vee C \equiv A \vee (B \vee C) & \\ (A \wedge A) \equiv A & (A \wedge B) \wedge C \equiv A \wedge (B \wedge C) & \\ A \vee B \equiv B \vee A & \neg(A \vee B) \equiv (\neg A) \wedge (\neg B) & \\ A \wedge B \equiv B \wedge A & \neg(A \wedge B) \equiv (\neg A) \vee (\neg B) & \\ (A \wedge \neg A) \equiv \mathbf{F} & A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C) & \\ (A \vee \neg A) \equiv \mathbf{T} & (A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C) & \\ (\mathbf{T} \wedge A) \equiv A & A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C) & \\ (\mathbf{T} \vee A) \equiv \mathbf{T} & (A \wedge B) \vee C \equiv (A \wedge C) \vee (B \wedge C) & \\ (\mathbf{F} \wedge A) \equiv \mathbf{F} & (\mathbf{F} \vee A) \equiv A & \end{array}$$

## Theorem

*Logical equivalence is a structural congruence. That is, if  $p \equiv p'$  and  $q \equiv q'$  then*

- ①  $\neg p \equiv \neg p'$
- ②  $p \wedge q \equiv p' \wedge q'$
- ③  $p \vee q \equiv p' \vee q'$
- ④  $p \Rightarrow q \equiv p' \Rightarrow q'$
- ⑤  $p \Leftrightarrow q \equiv p' \Leftrightarrow q'$

## Proof.

- Assume  $p \equiv p'$  and  $q \equiv q'$
- **Hyp:** Then for all valuations  $v$ ,  $v \models p$  iff  $v \models p'$  and  $v \models q$  iff  $v \models q'$ , i.e.  $\mathcal{I}_v(p) = \text{true}$  iff  $\mathcal{I}_v(p') = \text{true}$  and  $\mathcal{I}_v(q) = \text{true}$  iff  $\mathcal{I}_v(q') = \text{true}$
- Case 4: Show  $p \Rightarrow q \equiv p' \Rightarrow q'$ 
  - Other cases done same way
- Need to show for all  $v$ ,  $\mathcal{I}_v(p \Rightarrow q) = \text{true}$  iff  $\mathcal{I}_v(p' \Rightarrow q') = \text{true}$
- Fix  $v$
- Need to show if  $\mathcal{I}_v(p \Rightarrow q) = \text{true}$  then  $\mathcal{I}_v(p' \Rightarrow q') = \text{true}$ , and if  $\mathcal{I}_v(p' \Rightarrow q') = \text{true}$  then  $\mathcal{I}_v(p \Rightarrow q) = \text{true}$



## Proof.

- ( $\implies$ )
  - Assume  $\mathcal{I}_v(p \Rightarrow q) = \text{true}$
  - By closure property of inductive definition of  $\mathcal{I}$ , either  $\mathcal{I}_v(q) = \text{true}$  or  $\mathcal{I}_v(p) = \text{false}$ .
  - Therefore, by **Hyp**, either  $\mathcal{I}_v(q') = \text{true}$  or  $\mathcal{I}_v(p') = \text{false}$ 
    - since  $\mathcal{B}$  has only two elements, and  $\mathcal{I}_v$  total (proof?)
  - By  $\mathcal{I}$  def, have  $\mathcal{I}_v(p' \Rightarrow q')$
- ( $\impliedby$ ) □

# Non-standard Model of Propositional Logic

Other models possible

Example:

- $\mathcal{C} = \{\text{true}, \text{false}, \perp\}$
- Valuations assign values in  $\mathcal{C}$  to propositional atoms
- If  $\mathcal{J}_w(p) = \perp$  then  $\mathcal{J}_w(\neg p) = \perp$ , otherwise same as for  $\mathcal{I}$
- $\mathcal{J}_w(p) = \perp$  or  $\mathcal{J}_w(q) = \perp$  then  $\mathcal{J}_w(\neg p) = \perp$ ,  $\mathcal{J}_w(p \wedge q) = \perp$ ,  $\mathcal{J}_w(p \vee q) = \perp$ ,  $\mathcal{J}_w(p \Rightarrow q) = \perp$ , and  $\mathcal{J}_w(p \Leftrightarrow q) = \perp$ ; otherwise same as for  $\mathcal{I}$
- Note:  $A \vee \neg A \not\equiv \mathbf{T}$
  
- Other variants possible

# Proofs in Propositional Logic

- Natural Deduction proof is tree and a **discharge function**
  - Nodes are instances of inference rules
  - Leaves are assumptions of subproofs
  - Discharge function maps each leaf of the tree to an ancestor as prescribed by the inference rules

# Natural Deduction Inference Rules

- Inference rule has hypotheses and conclusion
- Conclusion a single proposition
- Hypotheses zero or more propositions, possibly with (discharged) hypotheses
- Rule with no hypotheses called an **axiom**
- Inference rule graphically presents as

$$\frac{H_1 \quad \dots \quad \begin{array}{c} A_i \\ \vdots \\ H_i \end{array} \quad \dots \quad H_j \quad \dots \quad \begin{array}{c} A_k \\ \vdots \\ H_k \end{array} \quad \dots \quad H_n}{C} \text{ rule}$$



# Natural Deduction Inference Rules

- Inference rules associated with connectives
- Two main kinds of inference rules:
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- Eliminations – says how to use a proposition made from connective to prove result
  - Example:

$$\frac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ Imp E}$$

# Introduction Rules

Truth Introduction:

$$\frac{}{\mathbf{T}} \text{ T I}$$

And Introduction:

$$\frac{A \quad B}{A \wedge B} \text{ And I}$$

Or Introduction:

$$\frac{A}{A \vee B} \text{ Or}_L \text{ I}$$

$$\frac{A}{B \vee A} \text{ Or}_R \text{ I}$$

Not Introduction:

$$\frac{\begin{array}{c} A \\ \vdots \\ \mathbf{F} \end{array}}{\neg A} \text{ Not I}$$

Implication Introduction:

$$\frac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \Rightarrow B} \text{ Imp I}$$

No False Introduction

# Example Proof 1

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$$A \Rightarrow (B \Rightarrow (A \wedge B))$$

# Example Proof 1

$$\frac{\frac{A}{\quad}}{B \Rightarrow (A \wedge B)} \quad \text{Imp I}$$
$$\frac{A \Rightarrow (B \Rightarrow (A \wedge B))}{\quad}$$

# Example Proof 1

$$\frac{\frac{A \quad B}{A \wedge B}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$



# Example Proof 1

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

# Example Proof 1

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

- All assumptions discharged; proof complete

# Example Proof 2

$$\overline{B \Rightarrow (A \wedge B)}$$

## Example Proof 2

$$\frac{\frac{B}{A \wedge B}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

# Example Proof 2

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

# Example Proof 2

$$\frac{\frac{A? \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

## Example Proof 2

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved “Assuming  $A$ , we have  $B \Rightarrow (A \wedge B)$ ”

# Discharging Hypothesis

$$\frac{}{A \Rightarrow (A \wedge A)}$$



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$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

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$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

- **Imp I** (and other rules discharging assumptions) may discharge multiple instance of hypothesis

# Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I} \qquad \frac{}{A \Rightarrow (B \Rightarrow A)}$$

- **Imp I** (and other rules discharging assumptions) may discharge multiple instance of hypothesis

# Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

$$\frac{\frac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

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# Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

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# Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

$$\frac{\frac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

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$$A \Rightarrow (A \vee B)$$

# Elimination Rules

- So far, have rules to “introduce” logical connectives into propositions
- No rules for how to “use” logical connectives
  - No assumptions with logical connectives
- Need “elimination” rules
- Example: Can’t prove

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

with what we have so far

- Elimination rules assume assumption with a connective; have general conclusion
  - Generally needs additional hypotheses



# Elimination Rules

False Elimination:

$$\frac{F}{C} \text{ F E}$$

Not Elimination:

$$\frac{\neg A \quad A}{C} \text{ Not E}$$

And Elimination:

$$\frac{A \wedge B \quad \begin{array}{c} A \\ \vdots \\ C \end{array}}{C} \text{ And}_L \text{ E}$$

$$\frac{A \wedge B \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ And}_R \text{ E}$$

Or Elimination:

$$\frac{A \vee B \quad \begin{array}{c} A \\ \vdots \\ C \end{array} \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ Or E}$$

Implication Elimination:

$$\frac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ Imp E}$$

# Example Proof 4

$$\overline{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

# Example Proof 4

$$\frac{\overline{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

# Example Proof 4

$$\frac{\frac{\overline{A \Rightarrow C}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

# Example Proof 4

$$\frac{\frac{\frac{C}{A \Rightarrow C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}$$

# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{\quad C}{\quad} \text{Imp E}}{C} \text{Imp I}}{A \Rightarrow C} \text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{\quad C}{\quad} \text{Imp E}}{C} \text{Imp I}}{A \Rightarrow C} \text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{C}{C}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I} \text{ Imp E}$$



# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{A} \quad \frac{\frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I} \text{ Imp E}$$

# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{A} \quad \frac{\frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I} \text{ Imp E}$$

# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{C} \text{ Imp E} \quad \frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I} \quad \frac{C}{A \Rightarrow C} \text{ Imp I}}{\frac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}$$

# Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{A} \quad \frac{\frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I} \text{ Imp E}$$

# Some Well-Known Derived Rules

Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B} \text{MP}$$

$$\frac{A \Rightarrow B \quad A \quad B}{B} \text{Imp E}$$

Left Conjunct

$$\frac{A \wedge B}{A} \text{AndL}$$

$$\frac{A \wedge B \quad A}{A} \text{And}_L \text{ E}$$

Right Conjunct

$$\frac{A \wedge B}{B} \text{AndR}$$

$$\frac{A \wedge B \quad A}{A} \text{And}_R \text{ E}$$

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$$(A \wedge B) \Rightarrow (A \vee B)$$