HW 3 – Interpretation and Proof of First Order Logic Formulae

CS 477 – Spring 2014
Revision 1.0

Assigned February 21, 2018
Due February 28, 2018, 9:00 pm
Extension 48 hours (20% penalty)

1 Change Log

1.0 Initial Release.

2 Objectives and Background

The purpose of this HW is to test your understanding of

- modeling and interpretation of first order logic formulae

Another purpose of HWs is to provide you with experience answering non-programming written questions of the kind you may experience on the midterm and final.

3 Turn-In Procedure

The pdf for this assignment (`hw3.pdf`) should be found in the `assignments/hw3/` subdirectory of your svn directory for this course. Your solution should be put in that same directory. Using your favorite tool(s), you should put your solution in a file named `hw3-submission.pdf`. If you have problems generating a pdf, please seek help from the course staff. Your answers to the following questions are to be submitted electronically from within `assignments/hw3/` subdirectory by committing the file as follows:

```
svn add hw3-submission.pdf
svn commit -m "Turning in hw3"
```

4 Problems

Each of the formulae in Problems 1-3 is over the signature

$$\mathcal{G} = (V = \{u, v, w, x, y, z\}, F = \{+\}, af = \{+ \mapsto 2;\}, R = \{=, <\}, ar = \{= \mapsto 2, < \mapsto 2\})$$

. The operator $+$ and the relations $=$ and $<$ will be written as infixed. For each of the formulae in Problems 1-3, give the following:

a. (2 pts) the list of free variables;

b. for the structure $\mathcal{S} = \{\mathcal{G}, \mathcal{D} = \mathbb{N}, \mathcal{F}, \phi, \mathcal{R}, \rho\}$ where $\phi(+)$ is normal addition, and where $\rho(=)$ is normal equality and $\rho(<)$ is normal less-than comparison. ($\mathbb{N}$ is the non-negative integers.)

(i) (3 pts) give an assignment for which the formula is valid and say why the assignment satisfies the formula, or say why none exists, and

(ii) (3 pts) give an assignment for which the formula is invalid and say why the assignment fails to satisfy the formula, or say why none is possible;

c. for the structure $\mathcal{S} = \{\mathcal{G}, \mathcal{D} = \mathbb{R}, \mathcal{F}, \phi, \mathcal{R}, \rho\}$ where $\phi(+)$ is multiplication, and where $\rho(=)$ is normal equality but where $\rho(<)(x, y) = (x^2 < y^2)$.

(i) (3 pts) give an assignment for which the formula is valid and say why the assignment satisfies the formula, or say why none exists, and

(ii) (3 pts) give an assignment for which the formula is invalid and say why the assignment fails to satisfy the formula, or say why none is possible;

1. $\exists u.\, \forall v.\, (u < x) \wedge ((u < v) \Rightarrow ((x = v) \vee (x < v)))$
Solution:

1.

(a) Free variables = $\{x\}$

(b)

i. Let $a = \{u \mapsto 0, v \mapsto 0, w \mapsto 0, x \mapsto 1, y \mapsto 0, z \mapsto 0\}$. The only value that matters is $a(x)$. The update to a that demonstrates the existential maps u to 0. Then we have $0 < 1$, and if $0 < v$, then either $1 = v$ or $1 < v$ since in the naturals, there is no number strictly in between 0 and 1.

ii. Let $a = \{u \mapsto 0, v \mapsto 0, w \mapsto 0, x \mapsto 0, y \mapsto 0, z \mapsto 0\}$. Again, the only value that matters is $a(x)$. Here, since the clause $u < 0$ is unsatisfiable, the formula is invalid for this assignment.

(c)

i. There does not exist an assignment satisfying this formula. As above, the only value of the assignment that matters is $a(x)$. There are basically two possibilities for $a(x)$: it could be 0 or it could be something else. If $a(x) = 0$, then there does not exist a value for u satisfying $a(u) < a(x)$, so the formula as a whole can not be satisfied. For the other case, assume $a(x) \neq 0$, and assume $(a(u))^2 < (a(x))^2$. Then $(a(u))^2 < ((|a(u)| + (|a(x)|))/2)^2$ but $a(x) \neq (|a(u)| + (|a(x)|))/2$ and $((|a(u)| + (|a(x)|))/2)^2 < (a(x))^2$.

ii. Since there does not exist an assignment satisfying this formula, it is invalid for every assignment. In particular, it is invalid for $a = \{u \mapsto 0, v \mapsto 0, w \mapsto 0, x \mapsto 0, y \mapsto 0, z \mapsto 0\}$ since there is no possible value for $a(u)$ such that $(a(u))^2 < 0 = (a(x))^2$.

2. $(u < v) \wedge (v < w) \Rightarrow ((\exists x.\, ((u < x) \wedge (x < v))) \wedge (\exists u.(v < u) \wedge (u < w)))$

Solution:

4.

(a) Free variables = $\{u, v, w\}$

(b)

i. Let $a = \{u \mapsto 0, v \mapsto 2, w \mapsto 4, x \mapsto 0, y \mapsto 0, z \mapsto 0\}$. Then we need to show $(0 < 2) \wedge (2 < 4) \Rightarrow ((\exists x.\, ((0 < x) \wedge (x < 2))) \wedge (\exists u.(2 < u) \wedge (u < 4)))(\exists x.\, \exists y.((0 < x) \wedge (x < 2) \wedge (2 < y) \wedge (y < 4)))$. For this we use the incremental assignments $\{x \mapsto 1\}$ for the first existemtial and $\{u \mapsto 3\}$ for the second. Then we need that, if $(0 < 2) \wedge (2 < 4)$, then $(0 < 1) \wedge (1 < 2)$ and $(2 < 3) \wedge (3 < 4)$, which are both true.

ii. To give an assignment for which the formula is invalid, let $a = \{u \mapsto 0, v \mapsto 1, w \mapsto 2, x \mapsto 0, y \mapsto 0, z \mapsto 0\}$. For the formula to be valid under this assignment, we would need $(0 < 1) \wedge (1 < 2) \Rightarrow (\exists x.\ ((0 < x) \wedge (x < 1))) \wedge ((\exists u.(1 < u) \wedge (< 2)))$. That is, given that $(0 < 1) \wedge (1 < 2)$ is true, we would need assignment increments a' and a'' such that $(0 < a'(x)) \wedge (a'(x) < 1)$ and $(1 < a''(u)) \wedge (a''(u) < 2)$. But in the naturals there is no value between 0 and 1 or 1 and 2, so no such assignments exist, and the formula is invalid under the assignment a.

(c)

i. For this part, the same valuation as used in (b.i) will work here. Let $a = \{u \mapsto 0, v \mapsto 2, w \mapsto 4, x \mapsto 0, y \mapsto 0, z \mapsto 0\}$. Then we need to show $(0^2 < 2^2) \wedge (2^2 < 4^2) \Rightarrow ((\exists x.\ ((0^2 < x^2) \wedge (x^2 < 2^2))) \wedge (\exists u.\ (2^2 < u^2) \wedge (u^2 < 4^2)))$. But again, for the existentials, we can use the incremental assignments $\{x \mapsto 1\}$ and $\{u \mapsto 3\}$. Then we need to see that $(0^2 < 1^2) \wedge (1^2 < 2^2)$ and $(2^2 < 3^2) \wedge (3^2 < 4^2)$, that is, that $(0 < 1) \wedge (1 < 4) \wedge (4 < 9) \wedge (9 < 16)$, which is true.

ii. In this case, it is not possible to invalidate this formula with any assignment. Suppose there were an assignment a for which the formula was invalid. Then, since $(a(u)^2 < a(v)^2) \wedge (a(v)^2 < a(w)^2) \Rightarrow ((\exists x.\ ((a(u)^2 < x^2) \wedge (x^2 < a(v)^2)) \wedge (\exists y.\ (a(v)^2 < y^2) \wedge (y^2 < a(w)^2))))$ is false, we would have to have $(a(u)^2 < a(v)^2) \wedge (a(v)^2 < a(w)^2)$ is true (or the implication would be true). But then we could use the incremental assignments $\{x \mapsto ((|a(u)| + |a(v)|)/2)\}$ and $\{u \mapsto ((|a(v)| + |a(w)|)/2)\}$. Then we have $a(u)^2 < (a(u)^2 + 2|a(u)a(v)| + a(v)^2)/4$ and $(a(u)^2 + 2|a(u)a(v)| + a(v)^2)/4 < a(v)^2$ and $a(v)^2 < (a(v)^2 + 2|a(v)a(w)| + a(w)^2)/4$ and $(a(v)^2 + 2|a(v)a(w)| + a(w)^2)/4 < a(w)^2$. Therefore, since I can always construct this incremental assignments for such an a, the existsentials are always valid, and so the original formula is also always valid, and so no assignment invalidating it is possible in this model.

3. $\forall x.\ \forall y.(((x < y) \vee (x = y)) \wedge ((y < x) \vee (x = y))) \Rightarrow (x = y)$

Solution:

4.

(a) Free variables = $\{\}$

(b) Since there are no free variables, either $\mathcal{S}$ models the formula, or it models the negation of the formula. In this case, this formula is stating the principal of antisymmetry for $<$, which is true in $\mathbb{N}$. Therefore, any assignment, and in particular $a = \{u \mapsto 0, v \mapsto 0, w \mapsto 0, x \mapsto 1, y \mapsto 0, z \mapsto 0\}$, will satisfy the formula, and none will invalidate it.

(c) Again, either $\mathcal{S}$ models the formula, or it models its negation. Antisymmetry is also valid in the reals, and is unaffected by being restricted to numbers that are squares. $\mathcal{S}$ models the formula; every assignment satisfies it; none invalidate it.

For Problems 4 and 5, give a proof in the sequent encoding of Natural Deduction for First order Logic of the given formulae. You may give your proofs as written proof trees, or you may do your proofs in Isabelle and submit a separate `hw3.thy` file containing the proofs. The same restictions as for MP1 apply, except that you are also allowed to use the introduction and elimination rules `allI`, `allE`, `exI` and `exE`.

4. (10pts) $(\forall x.\forall y.P(x) \Rightarrow Q(y)) \Rightarrow ((\exists x.P(x)) \Rightarrow (\forall y.Q(y)))$

Solution: Let
$$\begin{aligned}
\Gamma_1 &= \{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x)\}\\
\Gamma_2 &= \{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x),\ P(c)\}\\
\Gamma_3 &= \{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x),\ P(c),\ \forall y.P(c) \Rightarrow Q(y)\}\\
\Gamma_4 &= \{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x),\ P(c),\ \forall y.P(c) \Rightarrow Q(y),\ P(c) \Rightarrow Q(y)\}\\
\Gamma_5 &= \{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x),\ P(c),\ \forall y.P(c) \Rightarrow Q(y),\ P(c) \Rightarrow Q(y),\ Q(y)\}
\end{aligned}$$

Then

$$\dfrac{\dfrac{\text{Hyp}}{\Gamma_1 \vdash \exists x.P(x)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_2 \vdash \forall x.\forall y.P(x) \Rightarrow Q(y)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash \forall y.P(c) \Rightarrow Q(y)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_4 \vdash P(c) \Rightarrow Q(y)} \quad \dfrac{\text{Hyp}}{\Gamma_4 \vdash P(c)} \quad \dfrac{\text{Hyp}}{\Gamma_5 \vdash Q(y)}}{\Gamma_4 \vdash Q(y)}\,\textsf{Imp E}}{\Gamma_3 \vdash Q(y)}\,\textsf{All E}}{\Gamma_2 \vdash Q(y)}\,\textsf{All E}}{\dfrac{\dfrac{\{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x)\} \vdash Q(y)}{\{\forall x.\forall y.P(x) \Rightarrow Q(y),\ \exists x.P(x)\} \vdash \forall y.Q(y)}\,\textsf{All I}}{\dfrac{\{\forall x.\forall y.P(x) \Rightarrow Q(y)\} \vdash (\exists x.P(x)) \Rightarrow (\forall y.Q(y))}{\{\ \} \vdash (\forall x.\forall y.P(x) \Rightarrow Q(y)) \Rightarrow ((\exists x.P(x)) \Rightarrow (\forall y.Q(y)))}\,\textsf{Imp I}}\,\textsf{Imp I}}\,\textsf{Ex E}$$

5. (12 pts) $(\forall x.\forall y.P(x) \wedge P(y)) \Rightarrow ((\forall x.P(x)) \wedge (\forall y.P(y)))$

Solution: Let
$\Gamma_1 = \{\forall x.\forall y.P(x) \wedge P(y)\}$
$\Gamma_2 = \{\forall x.\forall y.P(x) \wedge P(y),\ \forall y.P(c) \wedge P(y)\}$
$\Gamma_3 = \{\forall x.\forall y.P(x) \wedge P(y),\ \forall y.P(c) \wedge P(y), P(c) \wedge P(c)\}$
$\Gamma_4 = \{\forall x.\forall y.P(x) \wedge P(y),\ \forall y.P(c) \wedge P(y),\ P(c) \wedge P(c),\ P(c)\}$
$\Gamma_5 = \{\forall x.\forall y.P(x) \wedge P(y),\ \forall y.P(d) \wedge P(y)\}$
$\Gamma_6 = \{\forall x.\forall y.P(x) \wedge P(y),\ \forall y.P(d) \wedge P(y), P(d) \wedge P(d)\}$
$\Gamma_7 = \{\forall x.\forall y.P(x) \wedge P(y),\ \forall y.P(d) \wedge P(y),\ P(d) \wedge P(d),\ P(d)\}$

and $LeftTree =$

$$\dfrac{\dfrac{\text{Hyp}}{\Gamma_1 \vdash \forall x.\forall y.P(x) \wedge P(y)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_2 \vdash \forall y.P(c) \wedge P(y)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash \forall y.P(c) \wedge P(c)} \quad \dfrac{\text{Hyp}}{\Gamma_4 \vdash P(c)}}{\Gamma_3 \vdash P(c)}\,\textsf{And}_L\ \textsf{E}}{\Gamma_2 \vdash P(c)}\,\textsf{All E}}{\Gamma_1 \vdash P(c)}\,\textsf{All E}$$

and $RightTree =$

$$\dfrac{\dfrac{\text{Hyp}}{\Gamma_1 \vdash \forall x.\forall y.P(x) \wedge P(y)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_5 \vdash \forall y.P(d) \wedge P(y)} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_6 \vdash \forall y.P(d) \wedge P(d)} \quad \dfrac{\text{Hyp}}{\Gamma_7 \vdash P(d)}}{\Gamma_6 \vdash P(d)}\,\textsf{And}_L\ \textsf{E}}{\Gamma_5 \vdash P(d)}\,\textsf{All E}}{\Gamma_1 \vdash P(d)}\,\textsf{All E}$$

Then

$$\dfrac{\dfrac{\dfrac{LeftTree}{\Gamma_1 \vdash \forall x.P(x)}\,\textsf{All I} \quad \dfrac{RightTree}{\Gamma_1 \vdash \forall y.P(y)}\,\textsf{All I}}{\{\forall x.\forall y.P(x) \wedge P(y)\} \vdash (\forall x.P(x)) \wedge (\forall y.P(y))}\,\textsf{And I}}{(\forall x.\forall y.P(x) \wedge P(y)) \Rightarrow ((\forall x.P(x)) \wedge (\forall y.P(y)))}\,\textsf{Imp I}$$

5 Extra Credit

6. (5 pts) Give a structure that models the formula in Problem 1 (different for either structure I gave) and describe why the structure models the formula.

   Solution: We need a domain, a binary function $+$, and two relations $<$ and $=$ in which every element has a left relative such that every right relative of the second is a left relative to the first by either the second relation or the first. One easy way to give a structure that models the formula id to have $+$ as a function that returns its first argument and to have each of the relations always returning true. Another example using more traditional meanings for $<$ and $=$ is to let the structure $\mathcal{S} = \{\mathcal{G}, \mathcal{D} = \mathbb{Z}, \mathcal{F}, \phi, \mathcal{R}, \rho\}$ where $\phi(+)$ is normal addition, and where $\rho(=)$ is normal equality and $\rho(<)$ is normal less-than comparison. ($\mathbb{Z}$ is all of the integers.) Then every integer has a unique immediate predecessor.