HW 1 – Truth and Proof in Propositional Logic

CS 477 – Spring 2018
Revision 1.0

Assigned January 24, 2018
Due January 31, 2018, 9:00 pm
Extension 48 hours (20% penalty)

1 Change Log

1.1 The extra credit problem has been revised to be only one of the two directions of the equivalence.

1.0 Initial Release.

2 Objectives and Background

The purpose of this HW is to test your understanding of

- validity of propositions in the standard model of propositional logic
- Natural Deduction proofs of propositions in propositional logic

Another purpose of HWs is to provide you with experience answering non-programming written questions of the kind you may experience on the midterm and final.

3 Turn-In Procedure

The pdf for this assignment (`hw1.pdf`) should be found in the `assignments/hw1/` subdirectory of your svn directory for this course. Your solution should be put in that same directory. Using your favorite tool(s), you should put your solution in a file named `hw1-submission.pdf`. If you have problems generating a pdf, please seek help from the course staff. Your answers to the following questions are to be submitted electronically from within `assignments/hw1/` subdirectory by commiting the file as follows:

```
svn add hw1-submission.pdf
svn commit -m "Turning in hw1"
```

4 Problem

For each of the following propositions, give both all possible valuations of every subformula of the proposition in the form of a truth table, and given a Natural Deduction proof of the proposition. For the Natural Deduction proof, you may use the pure style first indtruced in class, but it must be accompanied by a discription of how each assumption is discharged. Alternatively, you may use the sequent encoding of Natural Deduction proofs.

For the Natural Deduction proofs I have numbered the inferences in parentheses and I have given the discharge for each hypothesis in square brackets.

1. (5pts + 7pts) $(A \wedge B) \Rightarrow (B \wedge A)$

Solution:
Truth table:

A	B	$A \wedge B$	$B \wedge A$	$(A \wedge B) \Rightarrow (B \wedge A)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{A \wedge B\ [1] \quad B\ [4]}{B}\ \textsf{And}_R\ \textsf{E}\ (4) \qquad \dfrac{A \wedge B\ [1] \quad A\ [3]}{A}\ \textsf{And}_L\ \textsf{E}\ (3)}{B \wedge A}\ \textsf{And I}\ (2)}{(A \wedge B) \Rightarrow (B \wedge A)}\ \textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{A \wedge B\} \vdash A \wedge B}\ \textsf{Hyp} \qquad \dfrac{}{\{A \wedge B,\ B\} \vdash B}\ \textsf{Hyp}}{\{A \wedge B\} \vdash B}\ \textsf{And}_R\ \textsf{E} \qquad \dfrac{\dfrac{}{\{A \wedge B,\ B\} \vdash A \wedge B}\ \textsf{Hyp} \qquad \dfrac{}{\{A \wedge B,\ A\} \vdash A}\ \textsf{Hyp}}{\{A \wedge B\} \vdash A}\ \textsf{And}_L\ \textsf{E}}{\{A \wedge B\} \vdash B \wedge A}\ \textsf{And I}}{\{\ \} \vdash (A \wedge B) \Rightarrow (B \wedge A)}\ \textsf{Imp I}$$

2. (5pts + 6pts) $(A \vee A) \Rightarrow (B \vee A)$

Solution:
Truth table:

A	B	$A \vee A$	$B \vee A$	$(A \vee A) \Rightarrow (B \vee A)$
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{A \vee A\ [1] \quad A\ [3] \quad A\ [3]}{A}\ \textsf{Or E}\ (3)}{B \vee A}\ \textsf{Or}_\textsf{R}\ \textsf{I}\ (2)}{(A \vee A) \Rightarrow (B \vee A)}\ \textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\text{Hyp}}{\{A \lor A\} \vdash A \lor A} \qquad \cfrac{\text{Hyp}}{\{A \lor A,\ A\} \vdash A} \qquad \cfrac{\text{Hyp}}{\{A \lor A,\ A\} \vdash A}}{\{A \lor A\} \vdash A}\ \text{Or E}}{\{A \lor A\} \vdash B \lor A}\ \text{Or}_R\text{ I}}{\{\ \} \vdash (A \lor A) \Rightarrow (B \lor A)}\ \text{Imp I}$$

3. (7pts + 7pts) $(A \land B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$

Solution:
Truth table:

A	B	$A \land B$	$\neg A$	$\neg B$	$(\neg B) \Rightarrow (\neg A)$	$(A \land B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

Natural Deduction Proof:

$$\cfrac{\cfrac{\cfrac{A \land B\ [1] \qquad \cfrac{\neg B\ [2] \quad B\ [3]}{\neg A}\ \text{Not E}\ (4)}{\neg A}\ \text{And}_R\text{ E}\ (3)}{\neg B \Rightarrow \neg A}\ \text{Imp I}\ (2)}{(A \land B) \Rightarrow (\neg B \Rightarrow \neg A)}\ \text{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\text{Hyp}}{\{A \land B, \neg B\} \vdash A \land B} \qquad \cfrac{\cfrac{\text{Hyp}}{\{A \land B, \neg B, B\} \vdash \neg B} \quad \cfrac{\text{Hyp}}{\{A \land B, \neg B, B\} \vdash B}}{\{A \land B,\ \neg B,\ B\} \vdash \neg A}\ \text{Not E}}{\{A \land B, \neg B\} \vdash \neg A}\ \text{And}_R\text{ E}}{\{A \land B\} \vdash \neg B \Rightarrow \neg A}\ \text{Imp I}}{\{\ \} \vdash (A \land B) \Rightarrow (\neg B \Rightarrow \neg A)}\ \text{Imp I}$$

4. (7pts + 7pts) $(A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$

Solution:
Truth table:

A	B	$A \Rightarrow B$	$\neg A$	$\neg B$	$(\neg B) \Rightarrow (\neg A)$	$(A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg B\ [2] \qquad \dfrac{A \Rightarrow B\ [1] \quad A\ [3] \quad B\ [5]}{B}\ \textsf{Imp E}\ (5)}{F}\ \textsf{Not E}\ (4)}{\neg A}\ \textsf{Not I}\ (3)}{(\neg B) \Rightarrow (\neg A)}\ \textsf{Imp I}\ (2)}{(A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))}\ \textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textsf{Hyp}}{\{A \Rightarrow B, \neg B, A\} \vdash \neg B} \qquad \dfrac{\dfrac{\textsf{Hyp}}{\{A \Rightarrow B, \neg B, A\} \vdash A \Rightarrow B} \quad \dfrac{\textsf{Hyp}}{\{A \Rightarrow B, \neg B, A\} \vdash A} \quad \dfrac{\textsf{Hyp}}{\{A \Rightarrow B, \neg B, A, B\} \vdash B}}{\{A \Rightarrow B, \neg B, A\} \vdash B}\ \textsf{Imp E}}{\{A \Rightarrow B, \neg B, A\} \vdash F}\ \textsf{Not E}}{\{A \Rightarrow B, \neg B\} \vdash \neg A}\ \textsf{Not I}}{\{A \Rightarrow B\} \vdash (\neg B) \Rightarrow (\neg A)}\ \textsf{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))}\ \textsf{Imp I}$$

5. (8pts + 9pts) $((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

Solution:
Truth table:

A	B	C	$A \wedge B$	$(A \wedge B) \Rightarrow C$	$B \Rightarrow C$	$A \Rightarrow (B \Rightarrow C)$	$((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \wedge B) \Rightarrow C\ [1] \qquad \dfrac{A\ [2] \quad B\ [3]}{A \wedge B}\ \textsf{And I}\ (5) \qquad C\ [4]}{C}\ \textsf{Imp E}\ (4)}{B \Rightarrow C}\ \textsf{Imp I}\ (3)}{A \Rightarrow (B \Rightarrow C)}\ \textsf{Imp I}\ (2)}{((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))}\ \textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\{(A\wedge B)\Rightarrow C, A, B\}\vdash (A\wedge B)\Rightarrow C}\quad \dfrac{\dfrac{\text{Hyp}}{\{(A\wedge B)\Rightarrow C, A, B\}\vdash A}\quad \dfrac{\text{Hyp}}{\{(A\wedge B)\Rightarrow C, A, B\}\vdash B}}{\{(A\wedge B)\Rightarrow C, A, B\}\vdash A\wedge B}\,\textsf{And I}\quad \dfrac{\text{Hyp}}{\{(A\wedge B)\Rightarrow C, A, B, C\}\vdash C}}{\{(A\wedge B)\Rightarrow C, A, B\}\vdash C}\,\textsf{Imp E}}{\{(A\wedge B)\Rightarrow C, A\}\vdash B\Rightarrow C}\,\textsf{Imp I}}{\{(A\wedge B)\Rightarrow C\}\vdash A\Rightarrow (B\Rightarrow C)}\,\textsf{Imp I}}{\{\ \}\vdash ((A\wedge B)\Rightarrow C)\Rightarrow (A\Rightarrow (B\Rightarrow C))}\,\textsf{Imp I}$$

6. (7pts + 14pts) $((\neg B)\vee(\neg A))\Rightarrow(\neg(A\wedge B))$

Solution:
Truth table:

A	B	$\neg A$	$\neg B$	$(\neg B)\vee(\neg A)$	$A\wedge B$	$\neg(A\wedge B)$	$((\neg B)\vee(\neg A))\Rightarrow(\neg(A\wedge B))$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{(\neg B)\vee(\neg A)\ \ [1]\quad \dfrac{\neg B\ [3]\quad \dfrac{A\wedge B\ [2]\quad B\ [6]}{B}\,\textsf{And}_R\ \textsf{E}\ (6)}{F}\,\textsf{Not E}\ (4)\quad \dfrac{\neg A\ [3]\quad \dfrac{A\wedge B\ [2]\quad A\ [7]}{A}\,\textsf{And}_L\ \textsf{E}\ (7)}{F}\,\textsf{Not E}\ (5)}{F}\,\textsf{Or E}\ (3)}{\neg(A\wedge B)}\,\textsf{Not I}\ (2)}{((\neg B)\vee(\neg A))\Rightarrow(\neg(A\wedge B))}\,\textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:
Let $FalseFromNotB =$

$$\dfrac{\dfrac{\text{Hyp}}{\{(\neg B)\vee(\neg A), A\wedge B, \neg B\}\vdash \neg B}\quad \dfrac{\dfrac{\text{Hyp}}{\{(\neg B)\vee(\neg A), A\wedge B, \neg B\}\vdash A\wedge B}\quad \dfrac{\text{Hyp}}{\{(\neg B)\vee(\neg A), A\wedge B, \neg B, B\}\vdash B}}{\{(\neg B)\vee(\neg A), A\wedge B, \neg B\}\vdash B}\,\textsf{And}_R\ \textsf{E}}{\{(\neg B)\vee(\neg A), A\wedge B, \neg B\}\vdash F}\,\textsf{Not E}$$

and $FalseFromNotA =$

$$\dfrac{\dfrac{\textsf{Hyp}}{\begin{array}{l}\{(\neg B)\vee(\neg A),\\ \quad A\wedge B,\neg A\}\vdash \neg A\end{array}} \qquad \dfrac{\dfrac{\textsf{Hyp}}{\begin{array}{l}\{(\neg B)\vee(\neg A),\\ \quad A\wedge B,\neg A\}\vdash A\wedge B\end{array}} \qquad \dfrac{\textsf{Hyp}}{\begin{array}{l}\{(\neg B)\vee(\neg A),\\ \quad A\wedge B,\neg A, A\}\vdash A\end{array}}}{\{(\neg B)\vee(\neg A),A\wedge B,\neg A\}\vdash A}\,\textsf{And}_L\ \textsf{E}}{\{(\neg B)\vee(\neg A),A\wedge B,\neg A\}\vdash F}\,\textsf{Not E}$$

Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\textsf{Hyp}}{\{(\neg B)\vee(\neg A),A\wedge B\}\vdash(\neg B)\vee(\neg A)} \qquad FalseFromNotB \qquad FalseFromNotA}{\{(\neg B)\vee(\neg A),A\wedge B\}\vdash F}\,\textsf{Or E}}{\{(\neg B)\vee(\neg A)\}\vdash\neg(A\wedge B)}\,\textsf{Not I}}{\{\ \}\vdash((\neg B)\vee(\neg A))\Rightarrow(\neg(A\wedge B))}\,\textsf{Imp I}$$

7. (8pts + 14pts) $((\neg A)\vee(\neg B))\Rightarrow(\neg(A\wedge B))$

Solution:
As you may have guessed, this is not the problem I had intended to set, but the problem makes sense, so here it is.

Truth table:

A	B	$\neg A$	$\neg B$	$(\neg A)\vee(\neg B)$	$A\wedge B$	$\neg(A\wedge B)$	$((\neg A)\vee(\neg B))\Rightarrow(\neg(A\wedge B))$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{(\neg A)\vee(\neg B)\ \ [1] \qquad \dfrac{\neg A\ [3] \qquad \dfrac{A\wedge B\ [2] \quad A\ [6]}{A}\,\textsf{And}_L\ \textsf{E}\ (6)}{F}\,\textsf{Not E}\ (4) \qquad \dfrac{\neg B\ [3] \qquad \dfrac{A\wedge B\ [2] \quad B\ [7]}{B}\,\textsf{And}_L\ \textsf{E}\ (7)}{F}\,\textsf{Not E}\ (5)}{F}\,\textsf{Or E}\ (3)}{\neg(A\wedge B)}\,\textsf{Not I}\ (2)}{((\neg A)\vee(\neg B))\Rightarrow(\neg(A\wedge B))}\,\textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:
Let $FalseFromNotA =$

$$\dfrac{\dfrac{\textsf{Hyp}}{\begin{array}{l}\{(\neg A)\vee(\neg B),\\ \quad A\wedge B,\neg A\}\vdash \neg A\end{array}} \qquad \dfrac{\dfrac{\textsf{Hyp}}{\begin{array}{l}\{(\neg A)\vee(\neg B),\\ \quad A\wedge B,\neg A\}\vdash A\wedge B\end{array}} \qquad \dfrac{\textsf{Hyp}}{\begin{array}{l}\{(\neg A)\vee(\neg B),\\ \quad A\wedge B,\neg A, A\}\vdash A\end{array}}}{\{(\neg A)\vee(\neg B),A\wedge B,\neg A\}\vdash A}\,\textsf{And}_L\ \textsf{E}}{\{(\neg A)\vee(\neg B),A\wedge B,\neg A\}\vdash F}\,\textsf{Not E}$$

and $FalseFromNotB =$

$$\dfrac{\dfrac{\text{Hyp}}{\{(\neg A)\vee(\neg B),\ A\wedge B,\neg B\}\vdash \neg B} \qquad \dfrac{\dfrac{\text{Hyp}}{\{(\neg A)\vee(\neg B),\ A\wedge B,\neg B\}\vdash A\wedge B} \qquad \dfrac{\text{Hyp}}{\{(\neg A)\vee(\neg B),\ A\wedge B,\neg B,B\}\vdash B}}{\{(\neg A)\vee(\neg B),A\wedge B,\neg B\}\vdash B}\ \text{And}_R\ \text{E}}{\{(\neg A)\vee(\neg B),A\wedge B,\neg B\}\vdash F}\ \text{Not E}$$

Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\{(\neg A)\vee(\neg B),A\wedge B\}\vdash(\neg A)\vee(\neg B)} \qquad FalseFromNotA \qquad FalseFromNotB}{\{(\neg A)\vee(\neg B),A\wedge B\}\vdash F}\ \text{Or E}}{\{(\neg A)\vee(\neg B)\}\vdash\neg(A\wedge B)}\ \text{Not I}}{\{\ \}\vdash((\neg A)\vee(\neg B))\Rightarrow(\neg(A\wedge B))}\ \text{Imp I}$$

5 Extra Credit

8. (10 pts) Give a detailed, rigorous proof of the following:

   For all propositions P, if there exists a proof of the sequent $\{\ \}\vdash P$ in the sequent encoding of the Natural Deduction system, then there exists a fully discharged proof of P in the Natural Deduction system.

   You will want to prove a more general fact by induction on the structure or height of proofs.

Solution:

First, both the notion of Natural Deduction proof and the sequent encoding are defined as inductive relations. Such a definition is equivalent to the existence of a tree (of finite height) with nodes labeled by the name of the rule and the conclusions of the inferences (of the inductive relation), and with the branches pointing to nodes labeled by the hypotheses. A node in a tree may be identified with the path (a sequence of numbers saying which branch to take) from the root to the node. The root is identified with the empty sequence.

We must show that if there exists a proof of the sequent $\{\ \}\vdash P$ in the sequent encoding of the Natural Deduction system, then there exists a proof of P in the Natural Deduction system. To do so, we will prove a more general fact, namely, that if there exists a proof of the sequent $\Gamma\vdash P$ in the sequent calculus, then there exists a Natural Deduction proof of P such that every undischarged assumption of the Natural Deduction proof is an element of Γ. If we prove this, then we have proved the main result because in that instance, $\Gamma=\{\}$, and so P has a Natural Deduction proof with all assumptions discharged.

We shall proceed by induction on the structure of the sequent proof. Fix a sequent proof of $\Gamma\vdash P$. By the inductive hypothesis, we may assume that for each hypothesis $\Delta\vdash Q$ of the last inference of the sequent proof, we have a Natural deduction proof, all of whose undischarged assumptions are elements of Δ. We shall additionally by case analysis on what the last inference is. there are seven introduction rules and six elimination rules, making for a total of thirteen cases.

Case Hyp: If the last inference is an instance of Hyp, then $P\in\Gamma$. In this case, we can build the Natural Deduction proof with P as the only node, and as a leaf it is an undischarged assumption. In this proof all undischarged assumptions (P) are in Γ, as was to be shown.

Case T I: If the last inference in our sequent proof is an instance T I. Then $P=\mathbf{T}$. By Truth Introduction $\dfrac{}{\mathbf{T}}\ \text{T I}$, we have a Natural Deduction proof of P with no undischarged assumptions, and hence with all undischarged assumptions being elements of Γ.

Case And I**:** If the last inference in our sequent proof is an instance of And I, then there exist propositions Q and R such $P = Q \wedge R$ and there exist subproofs of the sequent proof of $\Gamma \vdash P$ for each of $\Gamma \vdash Q$ and $\Gamma \vdash R$. Then, by the inductive hypothesis, we have Natural Deduction proofs for each of Q and R such that for each proof all of its undischarged assumptions are elements of Γ. Combining these two proofs using the Natural Deduction rule And I for Q and R, we get a Natural Deduction proof of P ($= Q \wedge R$), and combining the two discharge functions yields a discharge function for the newly constructed larger Natural Deduction proof such that all undischarged assumptions are in Γ.

Case $\mathsf{Or_L}$ I**:** If the last inference in our sequent proof is an instance of $\mathsf{Or_L}$ I, then there exist propositions Q and R such $P = Q \wedge R$ and there exists a subproof for the sequent $\Gamma \vdash Q$. Then, by the inductive hypothesis, we have a Natural Deduction proof for Q such that all of its undischarged assumptions are elements of Γ. Using the Natural Deduction rule $\mathsf{Or_L}$ I for Q, complemented with R, we get a Natural Deduction proof of P ($= Q \vee R$), and extending the discharge function for the proof of Q to essentially the same one for the extended proof yields a discharge function for the newly constructed larger Natural Deduction proof such that all undischarged assumptions are in Γ.

Case $\mathsf{Or_R}$ I**:** This case is essentially the same as for $\mathsf{Or_L}$ I, but with a subproof for the right disjunct instead of the left disjunct, but I will repeat it here. If the last inference in our sequent proof is an instance of $\mathsf{Or_R}$ I, then there exist propositions Q and R such $P = Q \wedge R$ and there exists a subproof for the sequent $\Gamma \vdash R$. Then, by the inductive hypothesis, we have a Natural Deduction proof for R such that all of its undischarged assumptions are elements of Γ. Using the Natural Deduction rule $\mathsf{Or_R}$ I for R, complemented with Q, we get a Natural Deduction proof of P ($= Q \vee R$), and extending the discharge function for the proof of R to essentially the same one for the extended proof yields a discharge function for the newly constructed larger Natural Deduction proof such that all undischarged assumptions are in Γ.

Case Not I**:** If the last inference in the sequent proof of $\Gamma \vdash P$ is an instance of Not I, then there exists a proposition Q such that $P = \neg Q$ and there is a subproof of the sequent $\Gamma \cup \{Q\} \vdash \mathbf{F}$. Then, by the induction, there exists a Natural Deduction proof of $\mathbf{F}$, all of whose undischarged assumptions are elements of $\Gamma \cup \{Q\}$. Then we can extend this to a proof of P using the Natural Deduction rule Not I. However, the discharge function for the Natural Deduction proof of $\mathbf{F}$, if simply extended to the Natural Deduction proof of P, could admit undischarged assumptions that are elements of $\Gamma \cup \{Q\}$ and not just Γ. However, the rule Not I admits the discharge of any assumption of Q. Therefore, we can update the extended discharge function by mapping every undischarged occurrence of the assumption of Q to the last node in the extended proof, thus yielding a Natural Deduction proof, all of whose undischarged assumptions are elements of Γ.

Case Imp I**:** If the last inference in the sequent proof of $\Gamma \vdash P$ is an instance of Imp I, then there exist propositions Q and R such that $P = Q \Rightarrow R$ and there is a subproof of the sequent $\Gamma \cup \{Q\} \vdash R$. Then, by the induction, there exists a Natural Deduction proof of R, all of whose undischarged assumptions are elements of $\Gamma \cup \{Q\}$. Then we can extend this to a proof of $Q \Rightarrow R$ using the Natural Deduction rule Imp I. However, the discharge function for the Natural Deduction proof of R, if simply extended to the Natural Deduction proof of $Q \Rightarrow R$, could admit undischarged assumptions that are elements of $\Gamma \cup \{Q\}$ and not just Γ. Since the rule Imp I admits the discharge of any assumption of Q, we can update the extended discharge function by mapping every undischarged occurrence of the assumption of Q to the last node in the extended proof, thus yielding a Natural Deduction proof, all of whose undischarged assumptions are elements of Γ.

Case Not E**:** If the last inference in the sequent proof of $\Gamma \vdash P$ is an instance of Not E, then there exists a proposition Q such that there exist subproofs for each of $\Gamma \vdash \neg Q$ and $\Gamma \vdash Q$. By the inductive hypothesis, we have Natural Deduction proofs for each of $\neg Q$ and Q such that for each proof all of its undischarged assumptions are elements of Γ. Combining these two proofs using the Natural Deduction rule Not E for $\neg Q$ and Q, we get a Natural Deduction proof of P, and combining the two discharge functions yields a discharge function for the newly constructed larger Natural Deduction proof such that all undischarged assumptions are in Γ.

Case F E**:** If the last inference in or sequent proof is an instance of F E, then there exists a subproof for the sequent $\Gamma \vdash \mathbf{F}$. Then, by the inductive hypothesis, we have a Natural Deduction proof for $\mathbf{F}$ such that all of its undischarged assumptions are elements of Γ. Using the Natural Deduction rule F E, complemented with P, we get a Natural Deduction proof of P, and extending the discharge function for the proof of $\mathbf{F}$ to the extended proof yields a discharge function for larger Natural Deduction proof such that all undischarged assumptions are in Γ.

Case And_L E**:** If the last inference in the sequent proof is an instance of And_L E, then there exist propositions Q

and R and there exist subproofs for the sequents $\Gamma \vdash Q \wedge P$ and $\Gamma \cup \{Q\} \vdash P$. Then, by the inductive hypothesis, we have a Natural Deduction proof for $Q \wedge R$ such that all of its undischarged assumptions are elements of Γ, and a Natural Deduction proof of P with all of its assumptions in $\Gamma \cup \{Q\}$. Using the Natural Deduction rule $\mathsf{And}_L\ \mathsf{E}$ with these proofs, we get another Natural Deduction proof of P. However, simply extending the discharge functions for the proofs of $Q \wedge R$ and P to essentially the union for the extended proof of P yields a discharge function for the newly constructed larger Natural Deduction proof such that it discharges all assumptions in the left part of the tree that are not in Γ, but is only guaranteed to discharge all the assumptions in the left part of the tree that are in $\Gamma \cup \{Q\}$. However, the rule $\mathsf{And}_L\ \mathsf{E}$ admits discharging assumptions of Q from the right subproof, so we can update the extended-unioned discharge function by additionally having it discharge all undischarged occurrences of Q in the right subproof at the newly introduced last node. Then all remaining undischarged assumptions are in Γ.

Case And_R **E:** This is essentially the same as the $\mathsf{And}_L\ \mathsf{E}$ case, but with P proved form R instead of Q, but I will repeat the argument. If the last inference in the sequent proof is an instance of $\mathsf{And}_R\ \mathsf{E}$, then there exist propositions Q and R and there exist subproofs for the sequents $\Gamma \vdash Q \wedge P$ and $\Gamma \cup \{R\} \vdash P$. Then, by the inductive hypothesis, we have a Natural Deduction proof for $Q \wedge R$ such that all of its undischarged assumptions are elements of Γ, and a Natural Deduction proof of P with all of its assumptions in $\Gamma \cup \{R\}$. Using the Natural Deduction rule $\mathsf{And}_R\ \mathsf{E}$ with these proofs, we get another Natural Deduction proof of P. However, simply extending the discharge functions for the proofs of $Q \wedge R$ and P to essentially the union for the extended proof of P yields a discharge function for the newly constructed larger Natural Deduction proof such that it discharges all assumptions in the left part of the tree that are not in Γ, but discharges all the assumptions in the left part of the tree that are not in $\Gamma \cup \{R\}$. However, the rule $\mathsf{And}_R\ \mathsf{E}$ admits discharging assumptions of R from the right subproof, so we can update the extended-unioned discharge function by additionally having it discharge all undischarged occurrences of R in the right subproof at the newly introduced last node. Then all remaining undischarged assumptions are in Γ.

Case Imp E**:** If the last inference in the sequent proof is an instance of Imp E, then there exist propositions Q and R and there exist subproofs for the sequents $\Gamma \vdash Q \Rightarrow R$, $\Gamma \vdash Q$ and $\Gamma \cup \{R\} \vdash P$. Then, by the inductive hypothesis, we have a Natural Deduction proof for $Q \Rightarrow R$ and Q such that all of its undischarged assumptions are elements of Γ, and a Natural Deduction proof of P with all of its assumptions in $\Gamma \cup \{R\}$. Using the Natural Deduction rule Imp E with these proofs, we get another Natural Deduction proof of P. However, simply extending the discharge functions for the proofs of $Q \Rightarrow R$, Q and P to essentially the union for the extended proof of P yields a discharge function for the newly constructed larger Natural Deduction proof such that it discharges all assumptions not in Γ that are in the left and middle subproofs of the proof, but discharges all the assumptions in the right subproof tree that are not in $\Gamma \cup \{R\}$. However, the rule Imp E admits discharging assumptions of R from the right subproof, so we can update the extended-unioned discharge function by additionally having it discharge all undischarged occurrences of R in the right subproof at the newly introduced last node. Then all remaining undischarged assumptions are in Γ.

Case Or E**:** If the last inference in the sequent proof is an instance of Or E, then there exist propositions Q and R and there exist subproofs for the sequents $\Gamma \vdash Q \vee R$, $\Gamma \cup \{Q\} \vdash P$ and $\Gamma \cup \{R\} \vdash P$. Then, by the inductive hypothesis, we have a Natural Deduction proof for $Q \vee R$ such that all of its undischarged assumptions are elements of Γ, a Natural Deduction proof of P with all of its assumptions in $\Gamma \cup \{Q\}$, and a Natural Deduction proof of P with all of its assumptions in $\Gamma \cup \{R\}$. Using the Natural Deduction rule Or E with these proofs, we get another Natural Deduction proof of P. However, again, simply extending the discharge functions for the proof of $Q \vee R$, and the two proofs of P to essentially the union for the extended proof of P yields a discharge function for the newly constructed larger Natural Deduction proof such that it discharges all assumptions not in Γ that are in the left subproof, all the assumptions not in $\Gamma \cup \{Q\}$ in the middle subproof and all the assumptions in the right subproof tree that are not in $\Gamma \cup \{R\}$. However, the rule Or E admits discharging assumptions of Q from the middle subproof and assumptions of R from the right subproof, so we can update the extended-unioned discharge function by additionally having it discharge all undischarged occurrences of Q in the middle subproof and of R in the right subproof at the newly introduced last node. Then all remaining undischarged assumptions are in Γ.

□