CS477 Formal Software Dev Methods

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Labeled Transition System (LTS)

A labeled tranistion system (LTS) is a 4-tuple (Q, Σ, δ, I) where

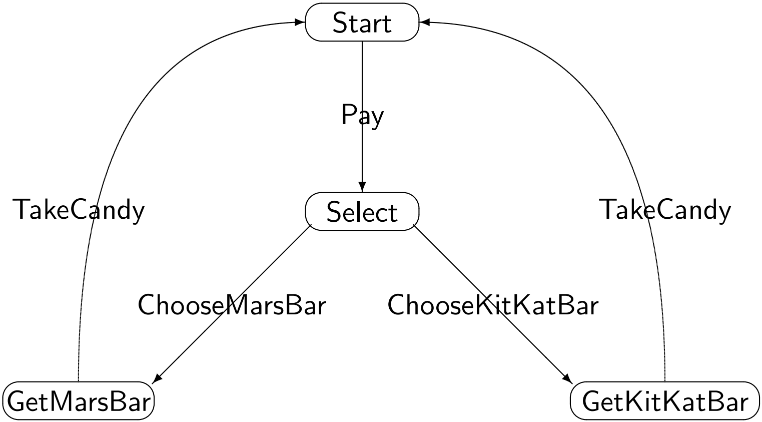
- Q set of states
  - Q finite or countably infinite
- Σ set of labels (aka actions)
  - Σ finite or countably infinite
- $\delta \subseteq Q \times \Sigma \times Q$ transition relation
- $I \subseteq Q$ initial states

Note: Write $q \xrightarrow{\alpha} q'$ for $(q, \alpha, q') \in \delta$.

Example: Candy Machine

- $Q = \{\text{Start}, \text{Select}, \text{GetMarsBar}, \text{GetKitKatBar}\}$
- $I = \{\text{Start}\}$
- $\Sigma = \{\text{Pay}, \text{ChooseMarsBar}, \text{ChooseKitKatBar}, \text{TakeCandy}\}$
- $\delta = \left\{ \begin{array}{l} (\text{Start}, \text{Pay}, \text{Select}) \\ (\text{Select}, \text{ChooseMarsBar}, \text{GetMarsBar}) \\ (\text{Select}, \text{ChooseKitKatBar}, \text{GetKitKatBar}) \\ (\text{GetMarsBar}, \text{TakeCandy}, \text{Start}) \\ (\text{GetKitKatBar}, \text{TakeCandy}, \text{Start}) \end{array} \right\}$

Example: Candy Machine



Predecessors, Successors and Determinism

Let (Q, Σ, δ, I) be a labeled transition system.

$$In(q, \alpha) = \{q' | q' \xrightarrow{\alpha} q\} \qquad In(q) = \bigcup_{\alpha \in \Sigma} In(q, \alpha)$$

$$Out(q, \alpha) = \{q' | q \xrightarrow{\alpha} q'\} \quad Out(q) = \bigcup_{\alpha \in \Sigma} Out(q, \alpha)$$

A labeled tranistion system (Q, Σ, δ, I) is deterministic if

$$|I| \leq 1 \text{ and } |Out(q, \alpha)| \leq 1$$

Labeled Transition Systems vs Finite State Automata

- LTS have no accepting states
  - Every FSA an LTS - just forget the accepting states
- Set of states and actions may be countably infinite
- May have infinite branching

Executions, Traces, and Runs

- A partial execution in an LTS is a finite or infinite alternating sequence of states and actions $\rho = q_0\alpha_1 q_1 \ldots \alpha_n q_n \ldots$ such that
  - $q_0 \in I$
  - $q_{i-1} \xrightarrow{\alpha_i} q_i$ for all i with q_i in sequence
- An execution is a maxial partial execution
- A finite or infinite sequence of actions $\alpha_1 \ldots \alpha_n \ldots$ is a trace if there exist states $q_0 \ldots q_n \ldots$ such that the sequence $q_0\alpha_1 q_1 \ldots \alpha_n q_n \ldots$ is a partial execution.
  - Let $\rho = q_0\alpha_1 q_1 \ldots \alpha_n q_n \ldots$ be a partial execution. Then $trace(\rho) = \alpha_1 \ldots \alpha_n \ldots$.

  A finite or inifnite sequence of states $q_0 \ldots q_n \ldots$ is a run if there exist actions $\alpha_1 \ldots \alpha_n \ldots$ such that the sequence $q_0\alpha_1 q_1 \ldots \alpha_n q_n \ldots$ is a partial execution.
  - Let $\rho = q_0\alpha_1 q_1 \ldots \alpha_n q_n \ldots$ be a partial execution. Then $run(\rho) = q_0 \ldots q_n \ldots$.

Example: Candy Machine

- Partial execution:
  $\rho = \mathit{Start} \cdot \mathit{Pay} \cdot \mathit{Select} \cdot \mathit{ChooseMarsBar} \cdot \mathit{GetMarsBar} \cdot \mathit{TakeCandy} \cdot \mathit{Start}$
- Trace: $\mathit{trace}(\rho) = \mathit{Pay} \cdot \mathit{ChooseMarsBar} \cdot \mathit{TakeCandy}$
- Run: $\mathit{run}(\rho) = \mathit{Start} \cdot \mathit{Select} \cdot \mathit{GetMarsBar} \cdot \mathit{Start}$

Program Transition System

A Program Transition System is a triple $(\mathcal{S}, T, init)$

- $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ is a first-order structure over signature $\mathcal{G} = (V, F, af, R, ar)$, used to interpret expressions and conditionals
- T is a finite set of conditional transitions of the form

$$g \to (v_1, \ldots, v_n) := (e_1, \ldots, e_n)$$

  where $v_i \in V$ distinct, and e_i term in $\mathcal{G}$, for $i = 1 \ldots n$

- $init$ initial condition asserted to be true at start of program

Example: Traffic Light

$V = \{Turn, NSC, EWC\}$, $F = \{NS, EW, Red, Yellow, Green\}$ (all arity 0), $R = \{=\}$

$$
\begin{array}{lrcl}
NSG & Turn = NS \wedge NSC = Red & \rightarrow & NSC := Green \\
NSY & Turn = NS \wedge NSC = Green & \rightarrow & NSC := Yellow \\
NSR & Turn = NS \wedge NSC = Yellow & \rightarrow & (Turn, NSC) := (EW, Red) \\
EWG & Turn = EW \wedge EWC = Red & \rightarrow & EWC := Green \\
EWY & Turn = EW \wedge EWC = Green & \rightarrow & EWC := Yellow \\
EWR & Turn = EW \wedge EWC = Yellow & \rightarrow & (Turn, EWC) := (NS, Red)
\end{array}
$$

$init = (NSC = Red \wedge EWC = Red \wedge (Turn = NS \vee Turn = EW)$

Mutual Exclusion (Attempt)

$$
\begin{array}{lll}
P1 :: & m1 : & \mathit{while\ true\ do} \\
& m2 : & \quad p11(*\mathit{not\ in\ crit\ sect}*) \\
& m3 : & \quad c1 := 0 \\
& m4 : & \quad \mathit{wait}(c2 = 1) \\
& m5 : & \quad r1(*\mathit{in\ crit\ sect}*) \\
& m6 : & \quad c1 := 1 \\
& m7 : & \mathit{od}
\end{array}
$$

$$
\begin{array}{lll}
P2 :: & n1 : & \mathit{while\ true\ do} \\
& n2 : & \quad p21(*\mathit{not\ in\ crit\ sect}*) \\
& n3 : & \quad c2 := 0 \\
& n4 : & \quad \mathit{wait}(c1 = 1) \\
& n5 : & \quad r2(*\mathit{in\ crit\ sect}*) \\
& n6 : & \quad c2 := 1 \\
& n7 : & \mathit{od}
\end{array}
$$

Mutual Exclusion PTS

$V = \{pc1, pc2, c1, c2\}$, $F = \{m1, \ldots, m6, n1, \ldots, n6, 0, 1\}$

$$
\begin{array}{rrcl}
T = & pc1 = m1 & \to & pc1 := m2 \\
& pc1 = m2 & \to & pc1 := m3 \\
& pc1 = m3 & \to & (pc1, c1) := (m4, 0) \\
& pc1 = m4 \wedge c2 = 1 & to & pc1 := m5 \\
& pc1 = m5 & \to & pc1 := m6 \\
& pc1 = m6 & \to & (pc1, c1) := (m1, 1) \\
& pc2 = n1 & \to & pc2 := n2 \\
& pc2 = n2 & \to & pc2 := n3 \\
& pc2 = n3 & \to & (pc2, c2) := (n4, 0) \\
& pc2 = n4 \wedge c1 = 1 & to & pc2 := n5 \\
& pc2 = n5 & \to & pc2 := n6 \\
& pc2 = n6 & \to & (pc2, c2) := (n1, 1)
\end{array}
$$

$init = (pc1 = m1 \wedge pc2 = n1 \wedge c1 = 1 \wedge c2 = 1)$

Interpreting PTS as LTS

Let $(\mathcal{S}, T, \mathit{init})$ be a program transition system. Assume V finite, $\mathcal{D}$ at most countable.

- Let $Q = V \to \mathcal{D}$, interpretted as all assingments of values to variables
  - Can restrict to mappings q where v and $q(v)$ have same type
- Let $\Sigma = T$
- Let $$\begin{aligned}\delta = \{(q, g \to (v_1, \ldots, v_n) := (e_1, \ldots, e_n), q') \mid \\ \mathcal{M}_q(g) \wedge \\ (\forall i \leq n . q'(v_i) = \mathcal{T}_q(e_i)) \wedge \\ (\forall v \notin \{v_1, \ldots, v_n\}.\ q'(v) = q(v))\}\end{aligned}$$
- $I = \{q | \mathcal{T}_q(\mathit{init}) = \mathbf{T}\}$

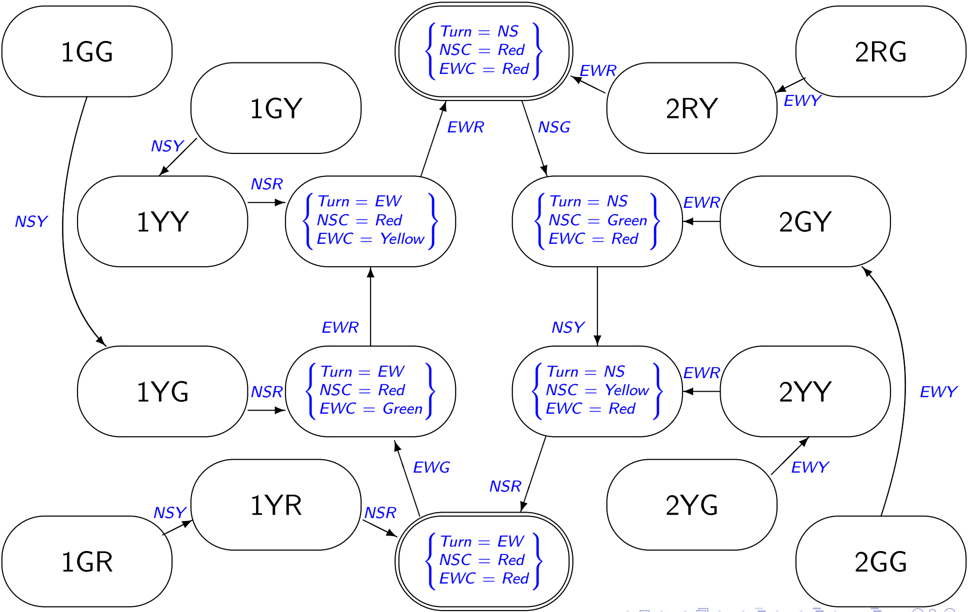
Example: Traffic Light

$V = \{Turn, NSC, EWC\}$, $F = \{NS, EW, Red, Yellow, Green\}$ (all arity 0), $R = \{=\}$

$$
\begin{array}{lrcl}
NSG & Turn = NS \wedge NSC = Red & \rightarrow & NSC := Green \\
NSY & NSC = Green & \rightarrow & NSC := Yellow \\
NSR & NSC = Yellow & \rightarrow & (Turn, NSC) := (EW, Red) \\
EWG & Turn = EW \wedge EWC = Red & \rightarrow & EWC := Green \\
EWY & EWC = Green & \rightarrow & EWC := Yellow \\
EWR & EWC = Yellow & \rightarrow & (Turn, EWC) := (NS, Red)
\end{array}
$$

$init = (NSC = Red \wedge EWC = Red \wedge (Turn = NS \vee Turn = EW)$

Example: Traffic Lights



Examples (cont)

- LTS for traffic light has $3 \times 3 \times 2 = 18$ possible well typed states
  - Is is possible to reach a state where $NSC \neq Red \wedge EWC \neq Red$ from an initial state?
  - If so, what sequence of actions allows this?
  - Do all the immediate predecessors of a state where $NSC = Green \vee EWC = Green$ satisfy $NSC = Red \wedge EWC = Red$?
  - If not, are any of those offend states reachable from and initial state, and if so, how?
- LTS for Mutual Exclusion has $6 \times 6 \times 2 \times 2 = 144$ posible well-tped states.
  - Is is possible to reach a state where $pc1 = m5 \wedge pc2 = n5$?
- How can we state these questions rigorously, formally?
- Can we find an algorithm to answer these types of questions?

Linear Temporal Logic - Syntax

$$\begin{array}{rcl}\varphi & ::= & p|(\varphi)|\ \not{\varphi}|\varphi \wedge \varphi'|\varphi \vee \varphi' \\ & | & \circ\varphi|\varphi\mathcal{U}\varphi'|\varphi\mathcal{V}\varphi'|\Box\varphi|\Diamond\varphi\end{array}$$

- p – a propostion over state variables
- $\circ\varphi$ – “next”
- $\varphi\mathcal{U}\varphi'$ – “until”
- $\varphi\mathcal{V}\varphi'$ – “releases”
- $\Box\varphi$ – “box”, “always”, “forever”
- $\Diamond\varphi$ – “diamond”, “eventually”, “sometime”

LTL Semantics: The Idea

$p \longrightarrow$	p													
$\circ\varphi \longrightarrow$		φ												
$\varphi\,\mathcal{U}\,\psi \longrightarrow$	φ	φ	φ	φ	φ	φ	φ	φ	ψ					
$\varphi\,\mathcal{V}\,\psi \longrightarrow$	ψ	ψ	ψ	ψ	ψ	ψ	φ, ψ							
$\Box\varphi \longrightarrow$	φ	φ	φ	φ	φ	φ	φ	φ	φ	φ	φ	φ	φ	φ
$\Diamond\varphi \longrightarrow$											φ			

Formal LTL Semantics

Given:

- $\mathcal{G} = (V, F, af, R, ar)$ signature expressing state propositions
- Q set of states,
- $\mathcal{M}$ modeling function over Q and $\mathcal{G}$: $\mathcal{M}(q, p)$ is true iff q models p. Write $q \models p$.
- $\sigma = q_0 q_1 \ldots q_n \ldots$ infinite sequence of state from Q.
- $\sigma^i = q_i q_{i+1} \ldots q_n \ldots$ the i^{th} tail of σ

Say σ models LTL formula φ, write $\sigma \models \varphi$ as follows:

- $\sigma \models p$ iff $q_0 \models p$
- $\sigma \models \neg\varphi$ iff $\sigma \not\models \varphi$
- $\sigma \models \varphi \wedge \psi$ iff $\sigma \models \varphi$ and $\sigma \models \psi$.
- $\sigma \models \varphi \vee \psi$ iff $\sigma \models \varphi$ or $\sigma \models \psi$.

Formal LTL Semantics

- $\sigma \models \circ\varphi$ iff $\sigma^1 \models \varphi$
- $\sigma \models \varphi \mathcal{U} \psi$ iff for some k, $\sigma^k \models \psi$ and for all $i < k$, $\sigma^i \models \varphi$
- $\sigma \models \varphi \mathcal{V} \psi$ iff for some k, $\sigma^k \models \varphi$ and for all $i \leq k$, $\sigma^i \models \psi$,
  or for all i, $\sigma^i \models \psi$.
- $\sigma \models \Box\varphi$ if for all i, $\sigma^i \models \psi$
- $\sigma \models \Diamond\varphi$ if for some i, $\sigma^i \models \psi$

Some Common Combinations

- $\Box\Diamond p$ "p will hold infinitely often"
- $\Diamond\Box p$ "p will continuously hold from some point on"
- $(\Box p) \Rightarrow (\Box q)$ "if p happens infinitely often, then so does q

Some Equivalences

- $\Box(\varphi \wedge \psi) = (\Box\varphi) \wedge (\Box\psi)$
- $\Diamond(\varphi \vee \psi) = (\Diamond\varphi) \vee (\Diamond\psi)$
- $\Box\varphi = \mathbf{F}\,\mathcal{V}\,\varphi$
- $\Diamond\varphi = \mathbf{T}\,\mathcal{U}\,\varphi$
- $\varphi\,\mathcal{V}\,\psi = \neg((\neg\varphi)\,\mathcal{U}\,(\neg\psi))$
- $\varphi\,\mathcal{U}\,\psi = \neg((\neg\varphi)\,\mathcal{V}\,(\neg\psi))$
- $\neg(\Diamond\varphi) = \Box(\neg\varphi)$
- $\neg(\Box\varphi) = \Diamond(\neg\varphi)$

Some More Eqivalences

- $\Box\varphi = \varphi \wedge \circ\Box\varphi$
- $\Diamond\varphi = \varphi \vee \circ\Diamond\varphi$
- $\varphi\,\mathcal{V}\,\psi = (\varphi \wedge \psi) \vee (\psi \wedge \circ(\varphi\,\mathcal{V}\,\psi))$
- $\varphi\,\mathcal{U}\,\psi = \psi \vee (\varphi \wedge \circ(\varphi\,\mathcal{V}\,\psi)$
- $\Box$, $\Diamond$, $\mathcal{U}$, $\mathcal{V}$ may all be understood recursively, by what they state about right now, and what they state about the future
- Caution: $\Box$ vs $\Diamond$, $\mathcal{U}$ vs $\mathcal{V}$ differ in there limit behavior

Traffic Light Example

Basic Behavior:

- $\Box((NSC = Red) \vee (NSC = Green) \vee (NSC = Yellow))$
- $\Box((NSC = Red) \Rightarrow ((NSC \neq Green) \wedge (NSC \neq Yellow))$
- Similarly for $Green$ and Red
- $\Box(((NCS = Red) \wedge \circ(NCS \neq Red)) \Rightarrow \circ(NCS = Green))$
- Same as $\Box((NCS = Red) \Rightarrow ((NCS = Red)\,\mathcal{U}\,(NCS = Green)))$
- $\Box(((NCS = Green) \wedge \circ(NCS \neq Green)) \Rightarrow \circ(NCS = Yellow))$
- $\Box(((NCS = Yellow) \wedge \circ(NCS \neq Yellow)) \Rightarrow \circ(NCS = Red))$
- Same for EWC

Traffic Light Example

Basic Safety

- $\Box((NSC = Red) \vee (EWC = Red))$
- $\Box(\ ((NSC = Red) \wedge (EWC = Red))\ \mathcal{V}$
  $((NSC \neq Green) \Rightarrow (\circ(NSC = Green))))$

Basic Liveness

- $(\Diamond(NSC = Red)) \wedge (\Diamond(NSC = Green)) \wedge (\Diamond(NSC = Yellow))$
- $(\Diamond(EWC = Red)) \wedge (\Diamond(EWC = Green)) \wedge (\Diamond(EWC = Yellow))$

Proof System for LTL

- First step: View $\varphi \mathcal{V} \psi$ as moacro: $\varphi \mathcal{V} \psi = \neg((\neg\varphi)\mathcal{U}(\neg\psi))$
- Second Step: Extend all rules of Prop Logic to LTL
- Third Step: Add one more rule: $\dfrac{\varphi}{\Box\varphi}$ Gen
- Fourth Step: Add a collection of axioms (a sufficient set of 8 exists)
  - A1: $\Box\varphi \Leftrightarrow \neg(\Diamond(\neg\varphi))$
  - A2: $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$
  - A3: $\Box\varphi \Rightarrow (\varphi \wedge \circ\Box\varphi)$
  - A4: $\circ\neg\varphi \Leftrightarrow \neg \circ \varphi$
  - A5: $\circ(\varphi \Rightarrow \psi) \Rightarrow (\circ\varphi \Rightarrow \circ\psi)$
  - A6: $\Box(\varphi \Rightarrow \circ\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$
  - A7: $\varphi \mathcal{U} \psi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \circ(\varphi \mathcal{V} \psi)$
  - A8: $\varphi \mathcal{U} \psi \Rightarrow \Diamond\psi$
- Result: a sound and relatively complete proof system
- Can implement in Isabelle in much the same way as we did Hoare Logic