CS477 Formal Software Dev Methods

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First Order Logic vs Propositional Logic

First Order Logic extends Propositional Logic with

- Non-boolean constants
- Variables
- Functions and relations (or predicates, more generally)
- Quantification of variables

Sample first order formula:

$$\forall x. \exists y. x < y \land y \le x + 1$$

Reference: Peled, Software Reliability Methods, Chapter 3

Signatures

Start with signature:

$$G = (V, F, af, R, ar)$$

- V a countably infinite set of variables
- F finite set of function symbols
- $af: F \to \mathbb{N}$ gives the *arity*, the number of arguments for each function Constant c is a function symbol of arity 0 (af(c) = 0)
- R finite set of relation symbols
- $ar: R \to \mathbb{N}$, the arity for each relation symbol
 - Assumes $= \in R$ and ar(=) = 2

Terms over Signature

Terms t are expressions built over a signature (V, F, af, R, ar)

$$t ::= v$$
 $v \in V$
 $\mid f(t_1, \dots, t_n) \quad f \in F \text{ and } n = af(f)$

- Example: add(1, abs(x)) where $add, abs, 1 \in F$; $x \in V$
- For constant c write c instead of c()
- Will write s + t instead of +(s, t)
 - Similarly for other common infixes (e.g. +, -, *, ...)

Structures

Meaning of terms starts with a structure:

$$S = (G, D, F, \phi, R, \rho)$$

where

- G = (V, F, af, R, ar) a signature,
- D and domain on interpretation
- \mathcal{F} set of functions over \mathcal{D} ; $\mathcal{F} \subseteq \bigcup_{n \ge 0} \mathcal{D}^n \to \mathcal{D}$
 - Note: $\mathcal F$ can contain elements of $\mathcal D$ since $\mathcal D=(\mathcal D^0 o\mathcal D)$
- $\phi: F \to \mathcal{F}$ where if $\phi(f) \in (\mathcal{D}^n \to \mathcal{D})$ then n = af(f)
- \mathcal{R} set of relations over \mathcal{D} ; $\mathcal{R} \subseteq \bigcup_{n \ge 1} \mathcal{P}(\mathcal{D}^n)$
- $\rho: R \to \mathcal{R}$ where if $\rho(r) \subseteq \mathcal{D}^n$ then n = ar(r)

Assignments

V set of variables, \mathcal{D} domain of interpretation

An assignment is a function $a: V \to \mathcal{D}$

Example:

$$V = \{w, x, y, z\}$$

$$a = \{ w \mapsto 3.14, x \mapsto -2.75, y \mapsto 13.9, z \mapsto -25.3 \}$$

• Assignment is a fixed association of values to variables; not "update-able"

Interpretation of Terms

Fix structure $S = (G, D, F, \phi, R, \rho)$ where G = (V, F, af, R, ar)

For given assignment $a: V \to \mathcal{D}$, the interpretation \mathcal{T}_a of a term t is defined by structural induction on terms:

- $\mathcal{T}_a(v) = a(v)$ for $v \in V$
- $\bullet \ \mathcal{T}_{a}(f(t_1,\ldots,t_n)) = \phi(f)(\mathcal{T}_{a}(t_1),\ldots,\mathcal{T}_{a}(t_n))$

Example of Interpretation

- $V = \{w, x, y, z\}, \mathcal{D} = \mathbb{R}$
- 1, add, $abs \in F$, constant 1, and functions (in \mathcal{F}) for addition and absolute value respectively
- $a = \{ w \mapsto 3.14, x \mapsto -2.75, y \mapsto 13.9, z \mapsto -25.3 \}$

$$\begin{split} \mathcal{T}_{a}(add(1,abs(x))) &= \left(\mathcal{T}_{a}(1)\right) + \left(\mathcal{T}_{a}(abs(x))\right) \\ &= 1.0 + \left(\mathcal{T}_{a}(abs(x))\right) \\ &= 1.0 + \left|\mathcal{T}_{a}(x)\right| \\ &= 1.0 + \left|a(x)\right| \\ &= 1.0 + \left|-2.75\right| \\ &= 1.0 + 2.75 \\ &= 3.75 \end{split}$$

First-Order Formulae

First-order formulae built from terms using relations, logical connectives, quantifiers:

$$\begin{array}{lll} \textit{form} ::= \mathsf{true} & | & \mathsf{false} \\ & | & r(t_1, \dots, t_n) & r \in R, \ t_i \ \mathsf{terms}, \ n = \mathsf{ar}(r) \\ & | & (\mathit{form}) \mid \neg \mathit{form} \\ & | & \mathit{form} \land \mathit{form} \\ & | & \mathit{form} \lor \mathit{form} \\ & | & \mathit{form} \Rightarrow \mathit{form} \\ & | & \forall v.\mathit{form} \\ & | & \exists v.\mathit{form} \end{array}$$

Note: Scope of quantifiers as far to right as possible

$$\forall x.(x > y) \land (2 > x)$$
 same as $\forall x.((x > y) \land (2 > x))$
not same as $(\forall x.(x > y)) \land (2 > x)$

Subformulae

- ullet A subformula of formula ψ is a formula that occurs in ψ
 - More rigorous definition by structural induction on formulae
 - \bullet ψ subformula of ψ
 - ullet Use proper subformula to exclude ψ
- Write $\bigwedge_{i=1,\ldots,n} \psi_i$ for $\psi_1 \wedge \ldots \wedge \psi_n$
 - ψ_i called a conjunct
- Write $\bigvee_{i=1,\ldots,n} \psi_i$ for $\psi_1 \vee \ldots \vee \psi_n$
 - ullet ψ_i called a disjunct

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

For given assignment $a:V\to\mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

For given assignment $a:V\to\mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

•
$$\mathcal{M}_a(\text{true}) = \mathbf{T}$$

$$\mathcal{M}_a(\mathrm{false}) = \mathbf{F}$$

Interpretation of Formulae

Fix structure $S = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a:V\to\mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\text{true}) = \mathbf{T}$ $\mathcal{M}_a(\text{false}) = \mathbf{F}$
- ullet $\mathcal{M}_a(r(t_1,\ldots,t_n))=\mathbf{T}$ if $(\mathcal{T}_a(t_1),\ldots,\mathcal{T}_a(t_n))\in
 ho(r)$ and $\mathcal{M}_a(r(t_1,\ldots,t_n)) = \mathbf{F} \text{ if } (\mathcal{T}_a(t_1),\ldots,\mathcal{T}_a(t_n)) \notin \rho(r)$

Interpretation of Formulae

Fix structure $S = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a: V \to \mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\text{true}) = \mathbf{T}$ $\mathcal{M}_a(\mathrm{false}) = \boldsymbol{\mathsf{F}}$
- $\mathcal{M}_a(r(t_1,\ldots,t_n)) = \mathbf{T}$ if $(\mathcal{T}_a(t_1),\ldots,\mathcal{T}_a(t_n)) \in \rho(r)$ and $\mathcal{M}_a(r(t_1,\ldots,t_n)) = \mathbf{F} \text{ if } (\mathcal{T}_a(t_1),\ldots,\mathcal{T}_a(t_n)) \notin \rho(r)$
- $\mathcal{M}_{a}((\psi)) = \mathcal{M}_{a}(\psi)$

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

For given assignment $a:V\to\mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\text{true}) = \mathbf{T}$ $\mathcal{M}_a(\text{false}) = \mathbf{F}$
- $\mathcal{M}_{\mathsf{a}}(r(t_1,\ldots,t_n)) = \mathbf{T}$ if $(\mathcal{T}_{\mathsf{a}}(t_1),\ldots,\mathcal{T}_{\mathsf{a}}(t_n)) \in \rho(r)$ and $\mathcal{M}_{a}(r(t_{1},\ldots,t_{n})) = \mathbf{F} \text{ if } (\mathcal{T}_{a}(t_{1}),\ldots,\mathcal{T}_{a}(t_{n})) \notin \rho(r)$
- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$
- $\mathcal{M}_a(\neg \psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg \psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

For given assignment $a:V\to\mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\text{true}) = \mathbf{T}$ $\mathcal{M}_a(\text{false}) = \mathbf{F}$
- ullet $\mathcal{M}_{\mathsf{a}}(r(t_1,\ldots,t_n)) = \mathbf{T}$ if $(\mathcal{T}_{\mathsf{a}}(t_1),\ldots,\mathcal{T}_{\mathsf{a}}(t_n)) \in
 ho(r)$ and $\mathcal{M}_{a}(r(t_{1},\ldots,t_{n}))= extbf{F} ext{ if } (\mathcal{T}_{a}(t_{1}),\ldots,\mathcal{T}_{a}(t_{n}))\notin
 ho(r)$
- $\bullet \ \mathcal{M}_{a}((\psi)) = \mathcal{M}_{a}(\psi)$
- $\bullet \ \mathcal{M}_{\mathsf{a}}(\neg \psi) = \mathsf{T} \ \text{if} \ \mathcal{M}_{\mathsf{a}}(\psi) = \mathsf{F} \ \text{and} \ \mathcal{M}_{\mathsf{a}}(\neg \psi) = \mathsf{F} \ \text{if} \ \mathcal{M}_{\mathsf{a}}(\psi) = \mathsf{T}$
- $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ and $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, \mathit{af}, R, \mathit{ar})$

For given assignment $a: V \to \mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\text{true}) = \mathbf{T}$ $\mathcal{M}_a(\text{false}) = \mathbf{F}$
- ullet $\mathcal{M}_{\sf a}(r(t_1,\ldots,t_n))={f T}$ if $(\mathcal{T}_{\sf a}(t_1),\ldots,\mathcal{T}_{\sf a}(t_n))\in
 ho(r)$ and $\mathcal{M}_{\mathsf{a}}(r(t_1,\ldots,t_n)) = \mathbf{F} \text{ if } (\mathcal{T}_{\mathsf{a}}(t_1),\ldots,\mathcal{T}_{\mathsf{a}}(t_n)) \notin \rho(r)$
- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$
- $\mathcal{M}_a(\neg \psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg \psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ and $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{F}$ otherwise
- $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ or $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $S = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a:V\to\mathcal{D}$, the interpretation \mathcal{M}_a of a formula ψ assigning a value in $\{T, F\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\mathrm{false}) = \mathbf{F}$ • $\mathcal{M}_a(\text{true}) = \mathbf{T}$
- ullet $\mathcal{M}_{\sf a}(r(t_1,\ldots,t_n))={f T}$ if $(\mathcal{T}_{\sf a}(t_1),\ldots,\mathcal{T}_{\sf a}(t_n))\in
 ho(r)$ and $\mathcal{M}_{\mathsf{a}}(r(t_1,\ldots,t_n)) = \mathbf{F} \ \mathsf{if} \ (\mathcal{T}_{\mathsf{a}}(t_1),\ldots,\mathcal{T}_{\mathsf{a}}(t_n)) \notin \rho(r)$
- $\bullet \ \mathcal{M}_{a}((\psi)) = \mathcal{M}_{a}(\psi)$
- $\mathcal{M}_a(\neg \psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg \psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ and $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{F}$ otherwise
- $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ or $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{F}$ otherwise
- $\mathcal{M}_{\mathsf{a}}(\psi_1\Rightarrow\psi_2)=\mathsf{T}$ if $\mathcal{M}_{\mathsf{a}}(\psi_1)=\mathsf{F}$ or $\mathcal{M}_{\mathsf{a}}(\psi_2)=\mathsf{T}$, and $\mathcal{M}_a(\psi_1 \Rightarrow \psi_2) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

Let

$$a + [v \mapsto d] (w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

Let

$$(a + [v \mapsto d])(w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

Interpretation of Formulae

 $\mathcal{M}_a(\forall v.\psi) = \mathbf{F}$ otherwise

and $\mathcal{M}_a(\forall v.\psi) = \mathbf{F}$ otherwise

Fix structure $S = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

 $a + [v \mapsto d] (w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$

• $\mathcal{M}_a(\forall v.\psi) = \mathbf{T}$ if for every $d \in \mathcal{D}$ we have $\mathcal{M}_{a+[v \mapsto d]}(\psi) = \mathbf{T}$, and

• $\mathcal{M}_a(\exists v.\psi) = \mathbf{T}$ if there exists $d \in \mathcal{D}$ such that $\mathcal{M}_{a+[v\mapsto d]}(\psi) = \mathbf{T}$,

Interpretation of Formulae

Fix structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where G = (V, F, af, R, ar)

Let

$$a + [v \mapsto d] (w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

 $\bullet \ \mathcal{M}_a(\forall v.\psi) = \mathbf{T} \text{ if for every } d \in \mathcal{D} \text{ we have } \mathcal{M}_{a+[v \mapsto d]}(\psi) = \mathbf{T} \text{, and}$ $\mathcal{M}_a(\forall v.\psi) = \mathbf{F}$ otherwise

Modeling First-order Formulae

Given structure $S = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

- ullet (S, \mathcal{M}) model for first-order language over signature \mathcal{G}
- ullet Truth of formulae in language over signature ${\cal G}$ depends on structure
- Assignment a models ψ , or a satisfies ψ , or $a \models^{\mathcal{S}} \psi$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- ψ is valid for \mathcal{S} if $a \models^{\mathcal{S}} \psi$ for some a.
- S is a model of ψ , written $\models^S \psi$ if every assignment for S satisfies ψ .
- ψ is valid, or a tautology if ψ valid for every mode. Write $\models \psi$
- ψ_1 logically equivalent to ψ_2 if for all structures $\mathcal S$ and assignments a, $a \models^{\mathcal{S}} \psi_1 \text{ iff } a \models^{\mathcal{S}} \psi_2$

Examples

Let

- Assignment $\{x \mapsto 0\}$ satisfies $\exists y.x < y$ valid in interval [0, 1]; assignment $\{x \mapsto 1\}$ doesn't
- ullet $\forall x. \exists y. x < y$ valid in $\mathbb N$ and $\mathbb R$, but not interval $[0,\ 1]$
- $(\exists x. \forall y. (y \le x)) \Rightarrow (\forall y. \exists x. (y \le x))$ tautology Why?

Sample Tautologies

All instances of propositional tautologies

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All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \le x)) \Rightarrow (\forall y. \exists x. (y \le x))$$

Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \le x)) \Rightarrow (\forall y. \exists x. (y \le x))$$

$$\models ((\forall x. \forall y. \psi) \Leftrightarrow (\forall y. \forall x. \psi))$$

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Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \le x)) \Rightarrow (\forall y. \exists x. (y \le x))$$

$$\models ((\forall x. \forall y. \psi) \Leftrightarrow (\forall y. \forall x. \psi))$$

$$\models ((\forall x.\psi) \Rightarrow (\exists x.\psi))$$

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Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \le x)) \Rightarrow (\forall y. \exists x. (y \le x))$$

$$\models ((\forall x. \forall y. \psi) \Leftrightarrow (\forall y. \forall x. \psi))$$

$$\models ((\forall x.\psi) \Rightarrow (\exists x.\psi))$$

$$\models (\forall x.\psi_1 \land \psi_2) \Leftrightarrow ((\forall x.\psi_1) \land (\forall x.\psi_2))$$

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Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \le x)) \Rightarrow (\forall y. \exists x. (y \le x))$$

$$\models ((\forall x. \forall y. \psi) \Leftrightarrow (\forall y. \forall x. \psi))$$

$$\models ((\forall x.\psi) \Rightarrow (\exists x.\psi))$$

$$\models (\forall x.\psi_1 \land \psi_2) \Leftrightarrow ((\forall x.\psi_1) \land (\forall x.\psi_2))$$

$$(\exists x.\psi_1 \wedge \psi_2) \Rightarrow ((\exists x.\psi_1) \wedge (\exists x.\psi_2))$$

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Free Variables: Terms

Informally: free variables of a expression are variables that have an occurrence in an expression that is not bound. Written $f_V(e)$ for expression e

Free variables of terms defined by structural induction over terms; written

- $fv(x) = \{x\}$
- $f_{V}(f(t_1,\ldots,t_n)=\bigcup_{i=1,\ldots,n}f_{V}(t_i)$

Note:

- Free variables of term just variables occurring in term; no bound
- No free variables in constants
- Example: $fv(add(1, abs(x))) = \{x\}$

Free Variables: Formulae

• $fv(true) = fv(false) = \{ \}$

• $fv(\forall v. \psi) = fv(\exists v. \psi) = (fv(\psi) \setminus \{v\})$

Variable occurrence at quantifier are binding occurrence

 $(\mathit{fv}(\psi_1) \cup \mathit{fv}(\psi_2))$

Free Variables, Assignments and Interpretation

Theorem

Assume given structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, term t over G, and a and b assignments. If for every $x \in fv(t)$ we have a(x) = b(x) then $\mathcal{T}_a(t) = \mathcal{T}_b(a)$.

Assume given structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, formula ψ over G, and a and b assignments. If for every $x \in fv(\psi)$ we have a(x) = b(x) then $\mathcal{M}_a(\psi) = \mathcal{M}_b(\psi).$

Syntactic Substitution versus Assignment Update

Defined by structural induction on formulae; uses fv on terms

 $\bullet \ fv(r(t_1, \dots, t_n)) = \bigcup_{i=1,\dots,n} fv(t_i)$ $\bullet \ fv(\psi_1 \wedge \psi_2) = fv(\psi_1 \vee \psi_2) = fv(\psi_1 \Rightarrow \psi_2) = fv(\psi_1 \Leftrightarrow \psi_2) = fv(\psi$

Occurrence that is not free and not binding is a bound occurrence

Example: $fv(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y))) = \{x, z\}$

- When interpreting universal quantification $(\forall x. \psi)$, wanted to check interpretation of every instance of ψ where v was replaced by element of semantic domain \mathcal{D}
- ullet How: semantically interpret ψ with assignment updated by $v\mapsto d$ for every $d \in \mathcal{D}$
- Syntactically?
- Answer: substitution

Substitution in Terms

- Substitution of term t for variable x in term s (written s[t/x]) gotten by replacing every instance of x in s by t
 - x called redex; t called residue
- Yields instance of s

Formally defined by structural induction on terms:

- $\bullet x[t/x] = t$
- y[t/x] = y for variable y where $y \neq x$
- $f(t_1,...,t_n)[t/x] = f(t_1[t/x],...,t_n[t/x])$

Example: (add(1, abs(x)))[add(x, y)/x] = add(1, abs(add(x, y)))

Substitution in Formulae: Problems

- Want to define by structural induction, similar to terms
- Quantifiers must be handled with care
 - Substitution only replaces free occurrences of variable Example:

$$(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y)))[x + 2/z] = (x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (x + 2 \ge y)))$$

• Need to avoid free variable capture **Example Problem:**

$$\begin{array}{l} (x>3 \land (\exists y.\ (\forall z.\ z \geq (y-x)) \lor (z \geq y)))[x+y/z] \neq \\ (x>3 \land (\exists y.\ (\forall z.\ z \geq (y-x)) \lor (x+y \geq y))) \end{array}$$

Theorem

Assume given structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, variable x, terms s and t over \mathcal{G} , and a assignment. Let $b = a[x \mapsto \mathcal{T}_a(t)]$. Then $\mathcal{T}_a(s[t/x]) = \mathcal{T}_b(s).$

Substitution in Formulae: Two Approaches

- When quantifier would capture free variable of redex, can't substitute in formula as is
- Solution 1: Make substitution partial function undefined in this case
- Solution 2: Define equivalence relation based on renaming bound variables; define substitution on equivalence classes
- Will take Solution 1 here
- Still need definition of equivalence up to renaming bound variables

Substitution in Formulae

- Defined by structural induction; uses substitution in terms
- Read equations below as saying left is not defined if any expression on right not defined
- true[t/x] = truefalse[t/x] = false
- $r(t_1,...,t_n)[t/x] = r((t_1[t/x],...,t_n[t/x]))$
- $(\psi)[t/x] = (\psi[t/x])$ $(\neg \psi)[t/x] = \neg(\psi[t/x])$
- $(\psi_1 \otimes \psi_2)[t/x] = (\psi_1[t/x]) \otimes (\psi_2[t/x])$ for $\emptyset \in \{\land, \lor, \Rightarrow, \Leftrightarrow\}$
- $\bullet \ (\mathcal{Q} \, x. \, \psi)[t/x] = \mathcal{Q} \, x. \, \psi \ \text{for} \ \mathcal{Q} \in \{\forall, \exists\}$
- $(Qy, \psi)[t/x] = Qy, (\psi[t/x])$ if $x \neq y$ and $y \notin fv(t)$ for $Q \in \{\forall, \exists\}$
- $(Qy, \psi)[t/x]$ not defined if $x \neq y$ and $y \in fv(t)$ for $Q \in \{\forall, \exists\}$

Substitution in Formulae

Examples

 $(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y)))[x + y/z]$ not defined

$$(x > 3 \land (\exists w. (\forall z. z \ge (w - x)) \lor (z \ge w)))[x + y/z] = (x > 3 \land (\exists w. (\forall z. z \ge (w - x)) \lor ((x + y) \ge y)))$$

Theorem

Assume given structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, formula ψ over G, and a assignment. If $\psi[t/x]$ defined, then $a \models^{S} \psi[t/x]$ if and only if $a[x \mapsto \mathcal{T}_a(t)] \models^{\mathcal{S}} \psi$

Renaming by Swapping: Terms

Define the swapping of two variables in a term $t[x \leftrightarrow y]$ by structural induction on terms:

- $x[x \leftrightarrow y] = y$ and $y[x \leftrightarrow y] = x$
- $z[x \leftrightarrow y] = z$ for z a variable, $z \neq x$, $z \neq y$
- $f(t_1,...,t_n)[x \leftrightarrow y] = f(t_1[x \leftrightarrow y],...,t_n[x \leftrightarrow y])$

Examples:

$$\begin{array}{ll} \operatorname{add}(1,\operatorname{abs}(\operatorname{add}(x,y)))[x \leftrightarrow y] &= \operatorname{add}(1,\operatorname{abs}(\operatorname{add}(y,x))) \\ \operatorname{add}(1,\operatorname{abs}(\operatorname{add}(x,y)))[x \leftrightarrow z] &= \operatorname{add}(1,\operatorname{abs}(\operatorname{add}(z,y))) \end{array}$$

Renaming by Swapping: Terms

Theorem

Assume given structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, variables x and y, term t over \mathcal{G} , and a assignment. Let $b = a[x \mapsto a(y)][y \mapsto a(x)]$. Then $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$

Renaming by Swapping: Terms

Proof.

By structural induction on terms, suffices to show theorem for the case where t variable, and case $t = f(t_1, \ldots, t_n)$, assuming result for t_1, \ldots, t_n

- Case: t variable
 - Subcase: t = x. Then $\mathcal{T}_a(x[x \leftrightarrow y]) = \mathcal{T}_a(y) = a(y)$ and $\mathcal{T}_b(x) = b(x) = a[x \mapsto a(y)][y \mapsto a(x)](x) = a[x \mapsto \mathcal{T}_a(y)](x) = a(y)$ so $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$
 - Subcase: t = y. Then $\mathcal{T}_a(y[x \leftrightarrow y]) = \mathcal{T}_a(x) = a(x)$ and $\mathcal{T}_b(y) = b(y) = a[x \mapsto a(y)][y \mapsto a(x)](x) = a(x)$ so $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$
 - Subcase: t = z variable, $z \neq x$ and $z \neq y$. Then Subcase. t = 2 variable, $2 \neq x$ and $2 \neq y$. Then $\mathcal{T}_a(z[x \leftrightarrow y]) = \mathcal{T}_a(z) = a(z) \text{ and}$ $\mathcal{T}_b(z) = b(z) = a[x \mapsto a(y)][y \mapsto a(x)](z) = a[x \mapsto \mathcal{T}_a(y)](z) = a(z)$ so $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$

Renaming by Swapping: Terms

Proof.

• Case: $t = f(t_1, \dots, t_n)$. Assume $\mathcal{T}_a(t_i[x \leftrightarrow y]) = \mathcal{T}_b(t_i)$ for $i = 1, \ldots, n$. Then

$$\begin{split} \mathcal{T}_{\text{a}}(t[x \leftrightarrow y]) &= \mathcal{T}_{\text{a}}(f(t_1, \dots, t_n)[x \leftrightarrow y]) \\ &= \mathcal{T}_{\text{a}}(f(t_1[x \leftrightarrow y], \dots, t_n[x \leftrightarrow y])) \\ &= \phi(f)(\mathcal{T}_{\text{a}}(t_1[x \leftrightarrow y]), \dots, \mathcal{T}_{\text{a}}(t_n[x \leftrightarrow y])) \\ &= \phi(f)(\mathcal{T}_{\text{b}}(t_1), \dots, \mathcal{T}_{\text{b}}(t_n)) \\ &= \inf_{\mathbf{x} \in \mathcal{T}_{\text{b}}(f(t_1, \dots, t_n))} \\ &= \mathcal{T}_{\text{b}}(f(t_1, \dots, t_n)) \\ &= \mathcal{T}_{\text{b}}(t) \quad \Box \end{split}$$

Renaming by Swapping: Formulae

Renaming by Swapping: Formulae

Define the swapping of two variables in a formula $\psi[x \leftrightarrow y]$ by structural induction, using swapping on terms:

- $\text{true}[x \leftrightarrow y] = \text{true}$ $false[x \leftrightarrow y] = false$
- $r(t_1,\ldots,t_n)[x\leftrightarrow y]=r((t_1[x\leftrightarrow y],\ldots,t_n[x\leftrightarrow y]))$
- $(\psi)[x \leftrightarrow y] = (\psi[x \leftrightarrow y])$ $(\neg \psi)[x \leftrightarrow y] = \neg(\psi[x \leftrightarrow y])$
- $(\psi_1 \otimes \psi_2)[x \leftrightarrow y] = (\psi_1[x \leftrightarrow y]) \otimes (\psi_2[x \leftrightarrow y])$ for $\otimes \in \{\land, \lor, \Rightarrow, \Leftrightarrow\}$
- $\bullet \ (\mathcal{Q} \, x. \, \psi)[x \leftrightarrow y] = \mathcal{Q} \, y. \, (\psi[x \leftrightarrow y]) \text{ for } \mathcal{Q} \in \{\forall, \exists\}$
- $(Qy.\psi)[x \leftrightarrow y] = Qy.(\psi[x \leftrightarrow y])$ for $Q \in \{\forall, \exists\}$
- $(Qz, \psi)[x \leftrightarrow y] = Qz, (\psi[x \leftrightarrow y])$ for z a variable with $z \neq x$, $z \neq y$, and $Q \in \{ \forall, \exists \}$

Examples

$$\begin{aligned} &(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y)))[x \leftrightarrow y] \\ &= (y > 3 \land (\exists x. (\forall z. z \ge (x - y)) \lor (z \ge x))) \\ &(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y)))[y \leftrightarrow z] \\ &(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y)))[y \leftrightarrow w] \end{aligned}$$

Theorem

Assume given structure $S = (G, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, variables x and y, formula ψ over \mathcal{G} , and a assignment. If $x \notin fv(t)$ and $y \notin fv(t)$ then $\psi[x \leftrightarrow y] \equiv \psi$

α -equivalence

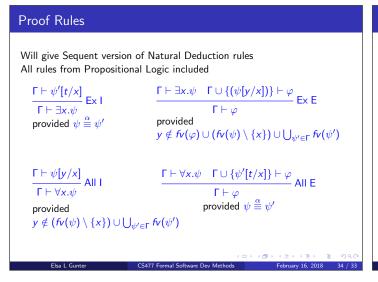
- If $\psi_1 \stackrel{\alpha}{\equiv} \psi_2$ then $\psi_2 \stackrel{\alpha}{\equiv} \psi$.
- It $\psi_1 \stackrel{\alpha}{\equiv} \psi_2$ and $\psi_2 \stackrel{\alpha}{\equiv} \psi_3$ then $\psi_1 \stackrel{\alpha}{\equiv} \psi_3$
- If $x \notin f_V(\psi)$ and $y \notin f_V(\psi)$ then $\psi \stackrel{\alpha}{=} \psi[x \leftrightarrow y]$.
- If $\psi_i \stackrel{\alpha}{=} \psi_i'$ for i = 1, 2 then
 - $(\psi_1) \stackrel{\alpha}{=} (\psi_1')$ $\neg \psi_1 \stackrel{\alpha}{=} \neg \psi_1'$
 - $\bullet \ \psi_1 \otimes \psi_2 \stackrel{\alpha}{\equiv} \psi_1' \otimes \psi_2' \text{ for } \otimes \in \{\land, \lor, \Rightarrow, \Leftrightarrow\}$
 - $\mathcal{Q}z. \psi_1 \stackrel{\alpha}{\equiv} \mathcal{Q}z. \psi_1'$ for $\mathcal{Q} \in \{ \forall, \exists \}$

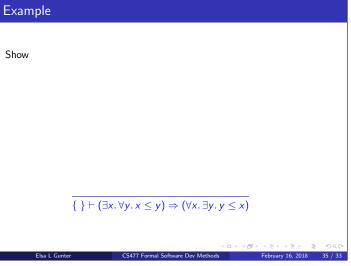
α -equivalence: Example

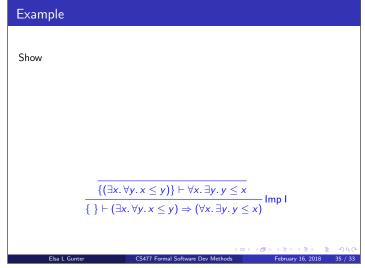
$$(x > 3 \land (\exists y. (\forall z. z \ge (y - x)) \lor (z \ge y)))$$

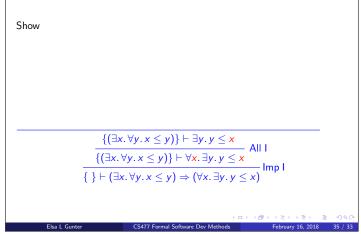
$$\stackrel{\alpha}{=} (x > 3 \land (\exists w. (\forall z. z \ge (w - x)) \lor (z \ge w)))$$

$$\begin{array}{l} (x > 3 \land (\exists y. \ (\forall z. \ z \ge (y - x)) \lor (z \ge y))) \\ \stackrel{\alpha}{=} (x > 3 \land (\exists w. \ (\forall y. \ y \ge (w - x)) \lor (z \ge w))) \end{array}$$





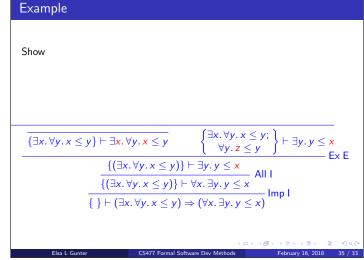


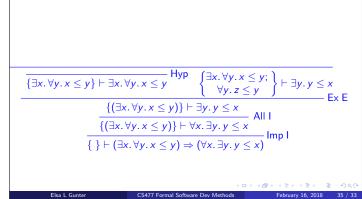


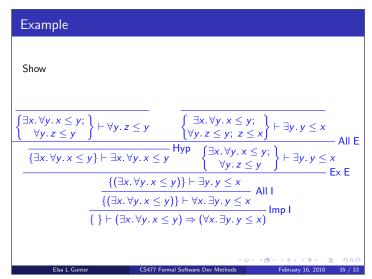
Example

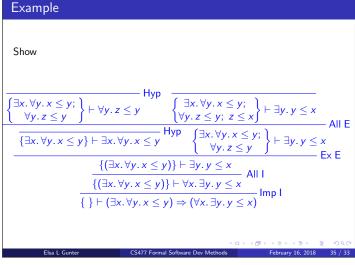
Example

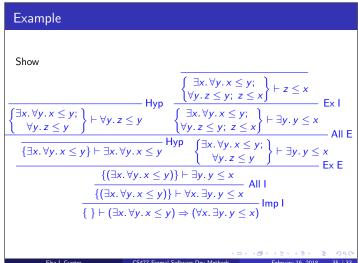
Show

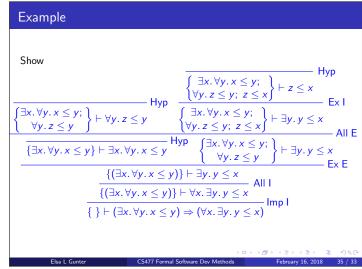


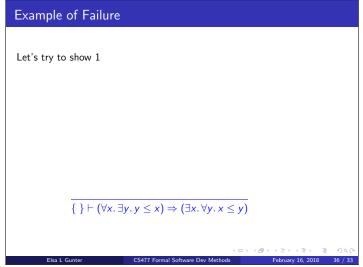


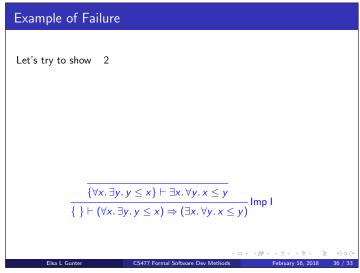












Example of Failure

Let's try to show 3

$$\frac{\overline{\{\forall x.\,\exists y.\,y\leq x\}\vdash \forall y.\,z\leq y}}{\{\forall x.\,\exists y.\,y\leq x\}\vdash \exists x.\,\forall y.\,x\leq y}\,\mathop{\rm Ex}\nolimits \, I}{\{\,\}\vdash (\forall x.\,\exists y.\,y\leq x)\Rightarrow (\exists x.\,\forall y.\,x\leq y)} \, Imp\, I$$

Example of Failure

Let's try to show

$$\frac{ \{ \forall x. \exists y. y \le x \} \vdash z \le x}{\{ \forall x. \exists y. y \le x \} \vdash \forall y. z \le y} \text{ All I}$$

$$\frac{ \{ \forall x. \exists y. y \le x \} \vdash \exists x. \forall y. x \le y}{\{ \forall x. \exists y. y \le x \} \vdash \exists x. \forall y. x \le y} \text{ Imp I}$$

Example of Failure

Let's try to show

Example of Failure

Let's try to show

$$\frac{\{\forall x. \exists y. y \le x\} \vdash \forall x. \exists y. y \le x}{\{\forall x. \exists y. y \le x\}} \vdash z \le x \\
\exists y. y \le x\} \vdash z \le x \\
\frac{\{\forall x. \exists y. y \le x\} \vdash z \le x}{\{\forall x. \exists y. y \le x\} \vdash \forall y. z \le y} \quad \text{All I} \\
\frac{\{\forall x. \exists y. y \le x\} \vdash \forall y. z \le y}{\{\forall x. \exists y. y \le x\} \vdash \exists x. \forall y. x \le y} \quad \text{Imp I}$$

Example of Failure

Let's try to show

$$\frac{\overline{Somethin}}{\{\forall x. \exists y. y \leq x\} \vdash \forall x. \exists y. y \leq x} \vdash z \leq x} + z \leq x$$

$$\frac{\{\forall x. \exists y. y \leq x\} \vdash \forall x. \exists y. y \leq x}{\{\forall x. \exists y. y \leq x\} \vdash z \leq x} \vdash z \leq x} \vdash z \leq x$$

$$\frac{\{\forall x. \exists y. y \leq x\} \vdash z \leq x}{\{\forall x. \exists y. y \leq x\} \vdash \forall y. z \leq y} \vdash z \leq x} \vdash z \leq x$$

$$\frac{\{\forall x. \exists y. y \leq x\} \vdash z \leq x}{\{\forall x. \exists y. y \leq x\} \vdash z \leq x} \vdash z \leq x} \vdash z \leq x$$

$$\frac{\{\forall x. \exists y. y \leq x\} \vdash z \leq x}{\{\forall x. \exists y. y \leq x\} \vdash z \leq x} \vdash z \leq x} \vdash z \leq x$$

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$$\frac{\{\forall x. \exists y. y \leq x\} \vdash z \leq x}{\{\forall x. \exists y. y \leq x\} \vdash z \leq x} \vdash z \leq x} \vdash z \leq x$$

$$\frac{\{\forall x. \exists y. y \leq x\} \vdash z \leq x}{\{\forall x. \exists y. y \leq x\} \vdash z \leq x} \vdash z \leq x} \vdash z \leq x$$

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$$\frac{\{\forall x. \exists y. y \leq x\} \vdash z \leq x}{\{\forall x. \exists y. y \neq x\} \vdash z \leq x} \vdash z \leq x$$

$$\frac{\{\forall x. \exists y. y \neq x\} \vdash z \leq x}{\{\forall x. \exists y. y \neq x\} \vdash z \leq x} \vdash z \leq x$$

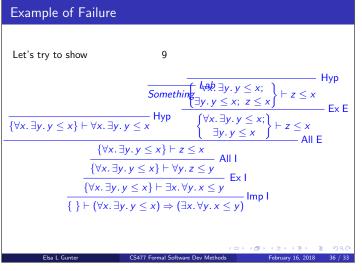
$$\frac{\{\forall x. \exists y. y \neq x\} \vdash z \leq x}{\{\forall x. \exists y. y \neq x\} \vdash z \leq x} \vdash z \leq x$$

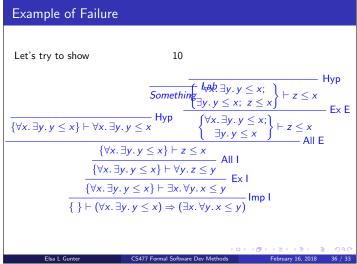
$$\frac{\{\forall x. \exists y. y \neq x\} \vdash z \leq x}{\{\forall x. \exists y. y \neq x\} \vdash z \leq x} \vdash z \leq x$$

$$\frac{\{\forall x. \exists y. y \neq x\} \vdash z \leq x}{\{\forall x. \exists y. y \neq x\} \vdash z \leq x} \vdash z \leq x$$

Example of Failure

Let's try to show





Floyd-Hoare Logic

- Also called Axiomatic Semantics
- Based on formal logic (first order predicate calculus)
- Logical system built from axioms and inference rules
- Mainly suited to simple imperative programming languages
- Ideas applicable quite broadly

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Floyd-Hoare Logic

 Used to formally prove a property (post-condition) of the state (the values of the program variables) after the execution of program, assuming another property (pre-condition) of the state holds before execution

Floyd-Hoare Logic

• Goal: Derive statements of form

- \bullet P, Q logical statements about state, P precondition, Q postcondition, C program
- Example:

$${x = 1} \ x := x + 1 \ {x = 2}$$

Floyd-Hoare Logic

• Approach: For each type of language statement, give an axiom or inference rule stating how to derive assertions of form

$$\{P\}$$
 C $\{Q\}$

where ${\it C}$ is a statement of that type

Compose axioms and inference rules to build proofs for complex programs

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Partial vs Total Correctness

- An expression $\{P\}$ (Q) is a partial correctness statement
- For total correctness must also prove that C terminates (i.e. doesnt run forever)
 - Written: [*P*] *C* [*Q*]
- Will only consider partial correctness here

Simple Imperative Language

• We will give rules for simple imperative language

```
\langle command \rangle ::= \langle variable \rangle := \langle term \rangle
  ⟨command⟩; ...; ⟨command⟩
  if \(\statement\) then \(\command\) else \(\command\)
 | while \langle statement \rangle do \langle command \rangle
```

• Could add more features, like for-loops

Substitution

• Notation: P[e/v] (sometimes $P[v \rightarrow e]$)

• Meaning: Replace every v in P by e

• Example:

$$(x+2)[y-1/x] = ((y-1)+2)$$

The Assingment Rule

$$\overline{\{P[e/x]\}\ x\ :=\ e\ \{P\}}$$

Example:

$$\overline{\{ ? \} x := y \{x = 2\}}$$

The Assingment Rule

$$\overline{\{P[e/x]\}\ x\ :=\ e\ \{P\}}$$

Example:

$$\overline{\{ | = 2 \} \ x := y \ \{x = 2 \}}$$

The Assingment Rule

$$\overline{\{P[e/x]\}\ x\ :=\ e\ \{P\}}$$

Example:

$$\{x = 2\} \ x := y \ \{x = 2\}$$

The Assingment Rule

$$\overline{\{P[e/x]\}\ x\ :=\ e\ \{P\}}$$

Examples:

$$\overline{\{y=2\}\ x\ :=\ y\ \{x=2\}}$$

$$\overline{\{y=2\}\ x\ :=\ 2\ \{y=x\}}$$

$$\overline{\{x+1=n+1\}\ x\ :=\ x+1\ \{x=n+1\}}$$

$$\{2=2\} \ x := 2 \ \{x=2\}$$

The Assignment Rule – Your Turn

• What is the weakest precondition of

$$x := x + y \{x + y = wx\}$$
?

$$\left\{ \begin{array}{c} ? \\ x := x + y \\ \left\{ x + y = wx \right\} \end{array} \right.$$

The Assignment Rule - Your Turn

• What is the weakest precondition of

$$x := x + y \{ x + y = wx \}?$$

{
$$(x+y) + y = w(x+y)$$
 }
 $x := x+y$
 $\{x+y = wx\}$

Precondition Strengthening

$$\frac{(P \Rightarrow P') \{P'\} \ C \ \{Q\}}{\{P\} \ C \ \{Q\}}$$

- Meaning: If we can show that P implies P' (i.e. $(P \Rightarrow P')$ and we can show that $\{P\}$ C $\{Q\}$, then we know that $\{P\}$ C $\{Q\}$
- P is stronger than P' means $P \Rightarrow P'$

Precondition Strengthening

• Examples:

$$\frac{x = 3 \Rightarrow x < 7 \quad \{x < 7\} \ x := x + 3 \ \{x < 10\}}{\{x = 3\} \ x := x + 3 \ \{x < 10\}}$$

$$\frac{\textit{True} \Rightarrow (2=2) \quad \{2=2\} \ x \ := \ 2 \ \{x=2\}}{\{\textit{True}\} \ x \ := \ 2 \ \{x=2\}}$$

$$\frac{x = n \Rightarrow x + 1 = n + 1}{\{x = n\}} \frac{\{x + 1 = n + 1\}}{\{x = n\}} \frac{x = x + 1}{\{x = n + 1\}}$$

Which Inferences Are Correct?

$$\frac{\{x > 0 \land x < 5\} \ x \ := \ x * x \ \{x < 25\}}{\{x = 3\} \ x \ := \ x * x \ \{x < 25\}}$$

$$\frac{\{x=3\}\ x\ :=\ x*x\ \{x<25\}}{\{x>0\land x<5\}\ x\ :=\ x*x\ \{x<25\}}$$

$$\frac{\{x * x < 25\} \ x := x * x \ \{x < 25\}}{\{x > 0 \land x < 5\} \ x := x * x \ \{x < 25\}}$$

Which Inferences Are Correct?

$$\frac{\{x > 0 \land x < 5\} \ x \ := \ x * x \ \{x < 25\}}{\{x = 3\} \ x \ := \ x * x \ \{x < 25\}} \ \textit{YES}$$

$$\frac{\{x=3\}\ x\ :=\ x*x\ \{x<25\}}{\{x>0\land x<5\}\ x\ :=\ x*x\ \{x<25\}}$$

$$\frac{\{x * x < 25\} \ x := x * x \ \{x < 25\}}{\{x > 0 \land x < 5\} \ x := x * x \ \{x < 25\}}$$

Which Inferences Are Correct?

$$\frac{\{x > 0 \land x < 5\} \ x \ := \ x * x \ \{x < 25\}}{\{x = 3\} \ x \ := \ x * x \ \{x < 25\}} \ \textit{YES}$$

$$\frac{\{x=3\}\ x\ :=\ x*x\ \{x<25\}}{\{x>0\land x<5\}\ x\ :=\ x*x\ \{x<25\}}\ \textit{NO}$$

$$\frac{\{x * x < 25\} \ x := x * x \ \{x < 25\}}{\{x > 0 \land x < 5\} \ x := x * x \ \{x < 25\}}$$

Which Inferences Are Correct?

$$\frac{\{x > 0 \land x < 5\} \ x \ := \ x * x \ \{x < 25\}}{\{x = 3\} \ x \ := \ x * x \ \{x < 25\}} \ \textit{YES}$$

$$\frac{\{x=3\}\ x\ :=\ x*x\ \{x<25\}}{\{x>0\land x<5\}\ x\ :=\ x*x\ \{x<25\}}\ \textit{NO}$$

$$\frac{\{x*x<25\}\ x\ :=\ x*x\ \{x<25\}}{\{x>0\land x<5\}\ x\ :=\ x*x\ \{x<25\}}\ \textit{YES}$$