HW 2 – Binary Decision Diagrams

CS 477 – Spring 2014

Revision 1.0

Assigned February 5, 2014
Due February 12, 2014, 9:00 pm
Extension 48 hours (20% penalty)

1 Change Log

1.0 Initial Release.

2 Objectives and Background

The purpose of this HW is to test your understanding of

- Binary Decision Diagrams and the Shannon Expansion

Another purpose of HWs is to provide you with experience answering non-programming written questions of the kind you may experience on the midterm and final.

3 Turn-In Procedure

The pdf for this assignment (`hw2.pdf`) should be found in the `assignments/hw2/` subdirectory of your svn directory for this course. Your solution should be put in that same directory. Using your favorite tool(s), you should put your solution in a file named `hw2-submission.pdf`. If you have problems generating a pdf, please seek help from the course staff. Your answers to the following questions are to be submitted electronically from within `assignments/hw2/` subdirectory by committing the file as follows:

```
svn add hw2-submission.pdf
svn commit -m "Turning in hw2"
```

4 Problem

For each of the following propositions,

a. (4 pts each) give the Shannon expansion (put it in if_then_else_ form),

b. (5 pts each) give the reduced ordered binary decision diagram (ROBDD), with the variables order smallest to largest alphabetically,

c. (5 pts each) give the reduced ordered binary decision diagram (ROBDD), with the variables order reverse alphabetically,

d. (3pts each) say whether it is satisfiable, and if it is, give a valuation satisfying it.

1. $(A \wedge B) \vee (A \wedge C)$

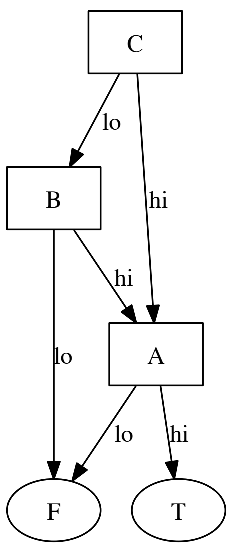
Solution:
I didn't specify and order on variables for Part a, so I will give the two most common. (Any of the six would be allowed.)

a.

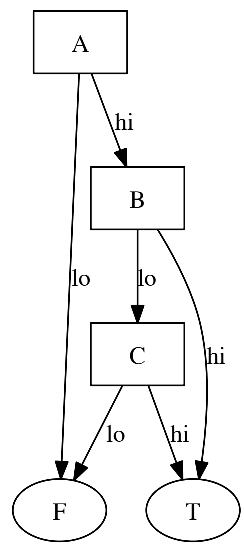
if A	then if B	then if C	then True
			else True
		else if C	then True
			else False
	else if B	then if C	then False
			else False
		else if C	then False
			else False

if C	then if B	then if A	then True
			else False
		else if A	then True
			else False
	else if B	then if A	then True
			else False
		else if A	then False
			else False

b. I intended $C > B > A$, and



c. $A > B > C$ but I excepted either interpretation.



d. I only asked for one valuation, but I'm giving a description of all here.
[$\{A \mapsto$ True; $B \mapsto$ True; $C \mapsto$ True$\}$;
$\{A \mapsto$ True; $B \mapsto$ True; $C \mapsto$ False$\}$;
$\{A \mapsto$ True; $B \mapsto$ False; $C \mapsto$ True$\}$]

2. $(A \Rightarrow (B \Rightarrow C)) \land ((A \Rightarrow B) \Rightarrow C)$

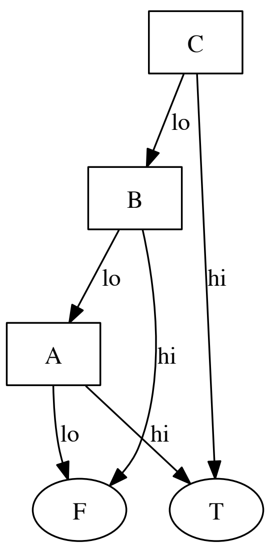
Solution:

a.

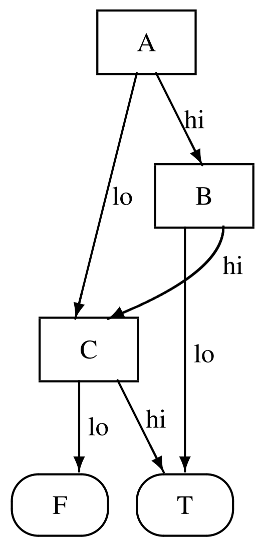
if A	then if B	then if C	then True
			else False
		else if C	then True
			else True
	else if B	then if C	then True
			else False
		else if C	then True
			else False

if C	then if B	then if A	then True
			else True
		else if A	then True
			else True
	else if B	then if A	then False
			else False
		else if A	then True
			else False

b. $C > B > A$



c. $A > B > C$



d. $[\ \{A \mapsto \text{True}; B \mapsto \text{True}; C \mapsto \text{True}\};$
$\{A \mapsto \text{True}; B \mapsto \text{False}; C \mapsto \text{True}\};$
$\{A \mapsto \text{True}; B \mapsto \text{False}; C \mapsto \text{False}\};$
$\{A \mapsto \text{False}; B \mapsto \text{True}; C \mapsto \text{True}\};$
$\{A \mapsto \text{False}; B \mapsto \text{False}; C \mapsto \text{True}\}\ ]$

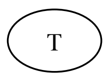
3. $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow (B \wedge C))$

Solution:

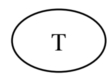
a.

if A	then if B	then if C	then True	if C	then if B	then if A	then True
			else True				else True
		else if C	then True			else if A	then True
			else True				else True
	else if B	then if C	then True		else if B	then if A	then True
			else True				else True
		else if C	then True			else if A	then True
			else True				else True

b. $C > B > A$



c. $A > B > C$



d.

$$
\begin{array}{l}
[\ \{A \mapsto \mathsf{True}; B \mapsto \mathsf{True}; C \mapsto \mathsf{True}\}; \\
\ \ \{A \mapsto \mathsf{True}; B \mapsto \mathsf{True}; C \mapsto \mathsf{False}\}; \\
\ \ \{A \mapsto \mathsf{True}; B \mapsto \mathsf{False}; C \mapsto \mathsf{True}\}; \\
\ \ \{A \mapsto \mathsf{True}; B \mapsto \mathsf{False}; C \mapsto \mathsf{False}\}; \\
\ \ \{A \mapsto \mathsf{False}; B \mapsto \mathsf{True}; C \mapsto \mathsf{True}\}; \\
\ \ \{A \mapsto \mathsf{False}; B \mapsto \mathsf{True}; C \mapsto \mathsf{False}\}; \\
\ \ \{A \mapsto \mathsf{False}; B \mapsto \mathsf{False}; C \mapsto \mathsf{True}\}; \\
\ \ \{A \mapsto \mathsf{False}; B \mapsto \mathsf{False}; C \mapsto \mathsf{False}\}\ ]
\end{array}
$$

That is, every assignment models $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow (B \wedge C))$; it is a tautology.

5 Extra Credit

4. (10 pts) Given a detailed, rigorous proof that any two different reduced ordered BDDS, over the same variables with the same orderings, there exists a valuation that satisfies one and not the other.

Solution:
I will show, the contrapositive, that two ROBDDs over the same set of variables with the same orderings, where the ROBDDs have the same set of valutions over those variables, are equal. The proof will proceed by induction on the number of variables in our set. Recall that the root of an ROBDD is always the greatest variable in the ROBDD and that the variables along a path are always ordered in descending order. Let $V = \{v_1, \ldots, v_n\}$ be the set of variables over which our ROBDDs are constructed, and fix the numbering such that $v_i < v_i + 1$ for all i, $1 < i < n - 1$.

A path in an ROBDD is a sequence of pairs of a (low,high) label and a node such that the out-edge from the root having the first label points to the first node, and for each label-node pair that is not the last, the out-edge of the node labeled by the label of the next pair points to the next node. A valuation φ satisfies an ROBDD if and only if there exists a path ending in a leaf labeled with True such that the first label is the value of φ on the variable labeling the root, and for each adjacent pair (l_i, n_i), (l_{i+1}, n_{i+1}), if the label of n_i is v, then $\varphi(v) = \mathsf{False}$ if $l_{i+1} = \mathsf{low}$ and $\varphi(v) = \mathsf{True}$ if $l_{i+1} = \mathsf{high}$. Notice that if B is an ROBDD with root label v and B_{low} and B_{high} are the sub-ROBDDs pointed to by the low and high labeled edges respectively from the root, and A is the set of valuations satisfying B, then the set of valuations satisfying B_{low} is $\{\varphi|\exists\psi \in A.(\psi(v) = \text{False}) \wedge (\forall x.x \neq v \Rightarrow \varphi(x) = \psi(x))\}$ and the set of valuations satisfying B_{high} is $\{\varphi|\exists\psi \in A.(\psi(v) = \text{True}) \wedge (\forall x.x \neq v \Rightarrow \varphi(x) = \psi(x))\}$.

Base case: if we have no variables, then each of our ROBDDs is satisfiable if and only if it is a singleton node labeled True. We are assuming the set of valuations satsifying one is the same as the set satifying the other. Either the set is empty (there are no valuations satisfying either), in which case they are both the singleton node labeled False, or the set of valuations consists of all valuations over our universe of variables, and both ROBDDs are the singleton node labeled True, and hence equal. (If our universe of variables is empty this set of valuations contains just the singleton function with domain the empty set.)

Inductive case: Assume we have that for all ordered sets of variables of size less than n, and all pairs of reduced ordered ROBDDs over those variables, if the set of valuations satisfying the first ROBDD equals the set of valuations satisfying the second ROBDD, then the two ROBDDs are equal. Let us fix a set of variables $v_1 < \ldots < v_n$, ROBDDs B_1 and B_2 over those variables, and assume that the set of valuations that satisfy B_1 is the same as the set that satisfies B_2. Let us call that set of valuations A. We will do a case analysis on the labels of the roots of B_1 and B_2.

Case: the root of B_1 is not labeled by x_n. Then B_1 is an ROBDD over $\{x_1, \ldots, x_{n-1}\}$, a smaller set of variables, and A is still the set of valuations satisfying B_1. If the root of B_2 is also not labeled by v_n, then B_2 is also an ROBDD over $\{x_1, \ldots, x_{n-1}\}$ and A is still the set of valuations satisfying $B_1 2$. By the inductive hypothesis then we have the $B_1 = B_2$.

Suppose then that the root of B_2 is x_n. Let B_{low} and B_{high} be the sub-ROBDDs pointed to by the low and high labeled edges respectively from the root of B_2. As observed above,the set of valuations satisfying B_{low} is $A_{\text{low}} = \{\varphi|\exists\psi \in A.(\psi(v) = \text{False}) \wedge (\forall x.x \neq v \Rightarrow \varphi(x) = \psi(x))\}$ and the set of valuations satisfying B_{high} is $A_{\text{high}} = \{\varphi|\exists\psi \in A.(\psi(v) = \text{True}) \wedge (\forall x.x \neq v \Rightarrow \varphi(x) = \psi(x))\}$. Since the root of B_1 is not labeled by x_n, for every assignment $\varphi \in A$, we have the assignment φ' is in A, where $\varphi'(x_n) = \neg\varphi(x_n)$ and $\varphi'(v) = \varphi(v)$ for all $v \neq x_n$. Therefore we have that $A_{\text{low}} = A_{\text{high}} = A$. Both B_{low} and B_{high} are ROBDDs over the smaller set of variables $\{x_1, \ldots, x_{n-1}\}$ and they have the same set of evaluations, namely A. Therefore, by the inductive hypothesis both B_{low} and B_{high} are equal. However, this violates the "reduced" condition of being an ROBDD, since the low and high edges of every node are required to point to different (non-isomorphic) BDDs. Therefore, it must have been that the label on the root of B_2 was not x_n after all.

By the same argument with B_1 and B_2 reversed, if the root of B_2 is not labeled by x_n, then the root of B_1 is not labeled by x_n, and they are both equal.

Case: both B_1 and B_2 have roots labeled by x_n. Let $B_{1,\text{low}}$ and $B_{1,\text{high}}$ be the sub-ROBDDs pointed to by the low and high labeled edges respectively from the root of B_1 and $B_{2,\text{low}}$ and $B_{2,\text{high}}$ be the sub-ROBDDs pointed to by the low and high labeled edges respectively from the root of B_2. Then the set of valuations that satisfies $B_{1,\text{low}}$ is $\{\varphi|\exists\psi \in A.(\psi(v) = \text{False}) \wedge (\forall x.x \neq v \Rightarrow \varphi(x) = \psi(x))\}$, which is the same set of valuations that satisfies $B_{2,\text{low}}$. Since $B_{1,\text{low}}$ and $B_{2,\text{low}}$ are both ROBDDs over the smaller variable set $\{x_1, \ldots, x_{n-1}\}$ and they have the same set of evaluations, by the inductive hypothesis they are equal. Similarly, we have that $B_{1,\text{high}}$ and $B_{2,\text{high}}$ are equal. Since the root nodes of B_1 and B_2 are both labeled by the same variable, and the low out edges from the root both point to the same ROBDD, and the high out edges from the root both point to the same ROBDD, the two ROBDDs themselves must be equal.