HW 1 – Truth and Proof in Propositional Logic

CS 477 – Spring 2014

Revision 1.0

Assigned January 29, 2014
Due February 5, 2014, 9:00 pm
Extension 48 hours (20% penalty)

1 Change Log

1.0 Initial Release.

2 Objectives and Background

The purpose of this HW is to test your understanding of

- validity of propositions in the standard model of propositional logic
- Natural Deduction proofs of propositions in propositional logic

Another purpose of HWs is to provide you with experience answering non-programming written questions of the kind you may experience on the midterm and final.

3 Turn-In Procedure

The pdf for this assignment (`hw1.pdf`) should be found in the `assignments/hw1/` subdirectory of your svn directory for this course. Your solution should be put in that same directory. Using your favorite tool(s), you should put your solution in a file named `hw1-submission.pdf`. If you have problems generating a pdf, please seek help from the course staff. Your answers to the following questions are to be submitted electronically from within `assignments/hw1/` subdirectory by commiting the file as follows:

```
svn add hw1-submission.pdf
svn commit -m "Turning in hw1"
```

4 Problem

For each of the following propositions, give both all possible valuations of every subformula of the proposition in the form of a truth table, and given a Natural Deduction proof of the proposition. For the Natural Deduction proof, you may use the pure style first indtruced in class, but it must be accompanied by a discription of how each assumption is discharged. Alternatively, you may use the sequent encoding of Natural Deduction proofs.

For the Natural Deduction proofs I have numbered the inferences in parentheses and I have given the discharge for each hypothesis in square brackets.

1. (5pts + 7pts) $(A \wedge B) \Rightarrow (B \wedge A)$

Solution:
Truth table:

A	B	$A \wedge B$	$B \wedge A$	$(A \wedge B) \Rightarrow (B \wedge A)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{A \wedge B\ [1] \quad B\ [4]}{B}\ \mathsf{And}_R\ \mathsf{E}\ (4) \qquad \dfrac{A \wedge B\ [1] \quad A\ [3]}{A}\ \mathsf{And}_L\ \mathsf{E}\ (3)}{B \wedge A}\ \mathsf{And\ I}\ (2)}{(A \wedge B) \Rightarrow (B \wedge A)}\ \mathsf{Imp\ I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{Hyp}}{\{A \wedge B\} \vdash A \wedge B} \qquad \dfrac{\mathsf{Hyp}}{\{A \wedge B,\ B\} \vdash B}}{\{A \wedge B\} \vdash B}\ \mathsf{And}_R\ \mathsf{E} \qquad \dfrac{\dfrac{\mathsf{Hyp}}{\{A \wedge B,\ B\} \vdash A \wedge B} \qquad \dfrac{\mathsf{Hyp}}{\{A \wedge B,\ A\} \vdash A}}{\{A \wedge B\} \vdash A}\ \mathsf{And}_L\ \mathsf{E}}{\{A \wedge B\} \vdash B \wedge A}\ \mathsf{And\ I}}{\{\ \} \vdash (A \wedge B) \Rightarrow (B \wedge A)}\ \mathsf{Imp\ I}$$

2. (5pts + 6pts) $(A \vee A) \Rightarrow (B \vee A)$

Solution:
Truth table:

A	B	$A \vee A$	$B \vee A$	$(A \vee A) \Rightarrow (B \vee A)$
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{A \vee A\ [1] \quad A\ [3] \quad A\ [3]}{A}\ \mathsf{Or\ E}\ (3)}{B \vee A}\ \mathsf{Or}_R\ \mathsf{I}\ (2)}{(A \vee A) \Rightarrow (B \vee A)}\ \mathsf{Imp\ I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\{A \vee A\} \vdash A \vee A} \quad \dfrac{\text{Hyp}}{\{A \vee A,\ A\} \vdash A} \quad \dfrac{\text{Hyp}}{\{A \vee A,\ A\} \vdash A}}{\{A \vee A\} \vdash A}\ \text{Or E}}{\{A \vee A\} \vdash B \vee A}\ \text{Or}_R\ \text{I}}{\{\ \} \vdash (A \vee A) \Rightarrow (B \vee A)}\ \text{Imp I}$$

3. (7pts + 7pts) $(A \wedge B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$

Solution:
Truth table:

A	B	$A \wedge B$	$\neg A$	$\neg B$	$(\neg B) \Rightarrow (\neg A)$	$(A \wedge B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{A \wedge B\ [1] \qquad \dfrac{\neg B\ [2] \quad B\ [3]}{\neg A}\ \text{Not E}\ (4)}{\neg A}\ \text{And}_R\ \text{E}\ (3)}{\neg B \Rightarrow \neg A}\ \text{Imp I}\ (2)}{(A \wedge B) \Rightarrow (\neg B \Rightarrow \neg A)}\ \text{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\{A \wedge B, \neg B\} \vdash A \wedge B} \qquad \dfrac{\dfrac{\text{Hyp}}{\{A \wedge B, \neg B, B\} \vdash \neg B} \quad \dfrac{\text{Hyp}}{\{A \wedge B, \neg B, B\} \vdash B}}{\neg A}\ \text{Not E}}{\{A \wedge B, \neg B\} \vdash \neg A}\ \text{And}_R\ \text{E}\ (3)}{\{A \wedge B\} \vdash \neg B \Rightarrow \neg A}\ \text{Imp I}}{\{\ \} \vdash (A \wedge B) \Rightarrow (\neg B \Rightarrow \neg A)}\ \text{Imp I}$$

4. (7pts + 7pts) $(A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$

Solution:
Truth table:

A	B	$A \Rightarrow B$	$\neg A$	$\neg B$	$(\neg B) \Rightarrow (\neg A)$	$(A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Natural Deduction Proof:

$$
\begin{prooftree}
\AxiomC{$\neg B$ [2]}
\AxiomC{$A \Rightarrow B$ [1]}
\AxiomC{A [4]}
\AxiomC{B [6]}
\RightLabel{Imp E (6)}
\TrinaryInfC{B}
\RightLabel{Not E (4)}
\BinaryInfC{F}
\RightLabel{Not I (3)}
\UnaryInfC{$\neg A$}
\RightLabel{Imp I (2)}
\UnaryInfC{$(\neg B) \Rightarrow (\neg A)$}
\RightLabel{Imp I (1)}
\UnaryInfC{$(A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$}
\end{prooftree}
$$

Sequent Style Natural Deduction Proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Hyp}
\UnaryInfC{$\{A \Rightarrow B, \neg B, A\} \vdash \neg B$}
\AxiomC{}
\RightLabel{Hyp}
\UnaryInfC{$\{A \Rightarrow B, \neg B, A\} \vdash A \Rightarrow B$}
\AxiomC{}
\RightLabel{Hyp}
\UnaryInfC{$\{A \Rightarrow B, \neg B, A\} \vdash A$}
\AxiomC{}
\RightLabel{Hyp}
\UnaryInfC{$\{A \Rightarrow B, \neg B, A, B\} \vdash B$}
\RightLabel{Imp E}
\TrinaryInfC{$\{A \Rightarrow B, \neg B, A\} \vdash B$}
\RightLabel{Not E}
\BinaryInfC{$\{A \Rightarrow B, \neg B, A\} \vdash F$}
\RightLabel{Not I}
\UnaryInfC{$\{A \Rightarrow B, \neg B\} \vdash \neg A$}
\RightLabel{Imp I}
\UnaryInfC{$\{A \Rightarrow B\} \vdash (\neg B) \Rightarrow (\neg A)$}
\RightLabel{Imp I}
\UnaryInfC{$\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((\neg B) \Rightarrow (\neg A))$}
\end{prooftree}
$$

5. (8pts + 9pts) $((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

Solution:
Truth table:

A	B	C	$A \wedge B$	$(A \wedge B) \Rightarrow C$	$B \Rightarrow C$	$A \Rightarrow (B \Rightarrow C)$	$((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Natural Deduction Proof:

$$
\begin{prooftree}
\AxiomC{$(A \wedge B) \Rightarrow C$ [1]}
\AxiomC{A [2]}
\AxiomC{B [3]}
\RightLabel{And I (5)}
\BinaryInfC{$A \wedge B$}
\AxiomC{C [4]}
\RightLabel{Imp E (4)}
\TrinaryInfC{C}
\RightLabel{Imp I (3)}
\UnaryInfC{$B \Rightarrow C$}
\RightLabel{Imp I (2)}
\UnaryInfC{$A \Rightarrow (B \Rightarrow C)$}
\RightLabel{Imp I (1)}
\UnaryInfC{$((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$}
\end{prooftree}
$$

Sequent Style Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\{(A \wedge B) \Rightarrow C, A, B\} \vdash (A \wedge B) \Rightarrow C} \quad \dfrac{\dfrac{\text{Hyp}}{\{(A \wedge B) \Rightarrow C, A, B\} \vdash A} \quad \dfrac{\text{Hyp}}{\{(A \wedge B) \Rightarrow C, A, B\} \vdash B}}{\{(A \wedge B) \Rightarrow C, A, B\} \vdash A \wedge B}\,\text{And I} \quad \dfrac{\text{Hyp}}{\{(A \wedge B) \Rightarrow C, A, B, C\} \vdash C}}{\{(A \wedge B) \Rightarrow C, A, B\} \vdash C}\,\text{Imp E}}{\{(A \wedge B) \Rightarrow C, A\} \vdash B \Rightarrow C}\,\text{Imp I}}{\{(A \wedge B) \Rightarrow C\} \vdash A \Rightarrow (B \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash ((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))}\,\text{Imp I}$$

6. (7pts + 14pts) $((\neg B) \vee (\neg A)) \Rightarrow (\neg(A \wedge B))$

Solution:
Truth table:

A	B	$\neg A$	$\neg B$	$(\neg B) \vee (\neg A)$	$A \wedge B$	$\neg(A \wedge B)$	$((\neg B) \vee (\neg A)) \Rightarrow (\neg(A \wedge B))$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{(\neg B) \vee (\neg A)\ \ [1] \quad \dfrac{\neg B\ [3] \quad \dfrac{A \wedge B\ [2] \quad B\ [6]}{B}\,\text{And}_R\text{ E }(6)}{F}\,\text{Not E }(4) \quad \dfrac{\neg A\ [3] \quad \dfrac{A \wedge B\ [2] \quad A\ [7]}{A}\,\text{And}_L\text{ E }(7)}{F}\,\text{Not E }(5)}{F}\,\text{Or E }(3)}{\neg(A \wedge B)}\,\text{Not I }(2)}{((\neg B) \vee (\neg A)) \Rightarrow (\neg(A \wedge B))}\,\text{Imp I }(1)$$

Sequent Style Natural Deduction Proof:
Let $FalseFromNotB =$

$$\dfrac{\dfrac{\text{Hyp}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg B\} \vdash \neg B} \quad \dfrac{\dfrac{\text{Hyp}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg B\} \vdash A \wedge B} \quad \dfrac{\text{Hyp}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg B\} \vdash B}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg B\} \vdash B}\,\text{And}_R\text{ E}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg B\} \vdash F}\,\text{Not E}$$

and $FalseFromNotA =$

$$\dfrac{\dfrac{\textsf{Hyp}}{\{(\neg B) \vee (\neg A),\ A \wedge B, \neg A\} \vdash \neg A} \qquad \dfrac{\dfrac{\textsf{Hyp}}{\{(\neg B) \vee (\neg A),\ A \wedge B, \neg A\} \vdash A \wedge B} \qquad \dfrac{\textsf{Hyp}}{\{(\neg B) \vee (\neg A),\ A \wedge B, \neg A, A\} \vdash A}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg A\} \vdash A}\ \textsf{And}_L\ \textsf{E}}{\{(\neg B) \vee (\neg A), A \wedge B, \neg A\} \vdash F}\ \textsf{Not E}$$

Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\textsf{Hyp}}{\{(\neg B) \vee (\neg A), A \wedge B\} \vdash (\neg B) \vee (\neg A)} \qquad FalseFromNotB \qquad FalseFromNotA}{\{(\neg B) \vee (\neg A), A \wedge B\} \vdash F}\ \textsf{Or E}}{\{(\neg B) \vee (\neg A)\} \vdash \neg(A \wedge B)}\ \textsf{Not I}}{\{\ \} \vdash ((\neg B) \vee (\neg A)) \Rightarrow (\neg(A \wedge B))}\ \textsf{Imp I}$$

7. (8pts + 14pts) $((\neg A) \vee (\neg B)) \Rightarrow (\neg(A \wedge B))$

Solution:
As you may have guessed, this is not the problem I had intended to set, but the problem makes sense, so here it is.

Truth table:

A	B	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$	$A \wedge B$	$\neg(A \wedge B)$	$((\neg A) \vee (\neg B)) \Rightarrow (\neg(A \wedge B))$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Natural Deduction Proof:

$$\dfrac{\dfrac{\dfrac{(\neg A) \vee (\neg B)\ \ [1] \qquad \dfrac{\neg A\ [3] \qquad \dfrac{A \wedge B\ [2] \quad A\ [6]}{A}\ \textsf{And}_L\ \textsf{E}\ (6)}{F}\ \textsf{Not E}\ (4) \qquad \dfrac{\neg B\ [3] \qquad \dfrac{A \wedge B\ [2] \quad B\ [7]}{B}\ \textsf{And}_L\ \textsf{E}\ (7)}{F}\ \textsf{Not E}\ (5)}{F}\ \textsf{Or E}\ (3)}{\neg(A \wedge B)}\ \textsf{Not I}\ (2)}{((\neg A) \vee (\neg B)) \Rightarrow (\neg(A \wedge B))}\ \textsf{Imp I}\ (1)$$

Sequent Style Natural Deduction Proof:
Let $FalseFromNotA =$

$$\dfrac{\dfrac{\textsf{Hyp}}{\{(\neg A) \vee (\neg B),\ A \wedge B, \neg A\} \vdash \neg A} \qquad \dfrac{\dfrac{\textsf{Hyp}}{\{(\neg A) \vee (\neg B),\ A \wedge B, \neg A\} \vdash A \wedge B} \qquad \dfrac{\textsf{Hyp}}{\{(\neg A) \vee (\neg B),\ A \wedge B, \neg A, A\} \vdash A}}{\{(\neg A) \vee (\neg B), A \wedge B, \neg A\} \vdash A}\ \textsf{And}_L\ \textsf{E}}{\{(\neg A) \vee (\neg B), A \wedge B, \neg A\} \vdash F}\ \textsf{Not E}$$

and $FalseFromNotB =$

$$\dfrac{\dfrac{\textsf{Hyp}}{\begin{array}{c}\{(\neg A)\vee(\neg B),\\ A\wedge B,\neg B\}\vdash\neg B\end{array}} \quad \dfrac{\dfrac{\textsf{Hyp}}{\begin{array}{c}\{(\neg A)\vee(\neg B),\\ A\wedge B,\neg B\}\vdash A\wedge B\end{array}} \quad \dfrac{\textsf{Hyp}}{\begin{array}{c}\{(\neg A)\vee(\neg B),\\ A\wedge B,\neg B\}\vdash B\end{array}}}{\{(\neg A)\vee(\neg B),A\wedge B,\neg B\}\vdash B}\,\textsf{And}_R\ \textsf{E}}{\{(\neg A)\vee(\neg B),A\wedge B,\neg B\}\vdash F}\,\textsf{Not E}$$

Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\textsf{Hyp}}{\{(\neg A)\vee(\neg B),A\wedge B\}\vdash(\neg A)\vee(\neg B)} \quad FalseFromNotA \quad FalseFromNotB}{\{(\neg A)\vee(\neg B),A\wedge B\}\vdash F}\,\textsf{Or E}}{\{(\neg A)\vee(\neg B)\}\vdash\neg(A\wedge B)}\,\textsf{Not I}}{\{\ \}\vdash((\neg A)\vee(\neg B))\Rightarrow(\neg(A\wedge B))}\,\textsf{Imp I}$$

5 Extra Credit

8. (10 pts) Given a detailed, rigorous proof that the sequent encoding of Natural Deduction Proof System given in class is equivalent to Natural Deduction Proof System given in class for propositional logic in the following sense: For all propositions P, there exists a proof of P in the Natural Deduction system if and only if there exists a proof of the sequent $\{\ \}\vdash P$ in the sequent encoding of the Natural Deduction system.

Solution:

First, both the notion of Natural Deduction proof and the sequent encoding are defined as inductive relations. Such a definition is equivalent to the existence of a tree (of finite height) with nodes labeled by the conclusions of the inferences (of the inductive relation), and with the branches pointing to nodes labeled by the hypotheses. A node in a tree may be identified with the path (a sequence of numbers saying which branch to take) from the root to the node. The root is identified with the empty sequence.

There are two halves to this proof. We must show that if there exists a proof of P in the Natural Deduction system, then there exists a there exists a proof of the sequent $\{\ \}\vdash P$ in the sequent encoding of the Natural Deduction system, and if there exists a proof of the sequent $\{\ \}\vdash P$ in the sequent encoding of the Natural Deduction system, then there exists a proof of P in the Natural Deduction system.

I haven't finished typesetting this, but I wanted to get this up now so that the rest of it would be available.