# CS477 Formal Software Development Methods

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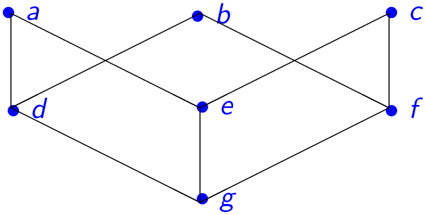
# Partial Orders

A partial order on a set $S$ is a binary relation $\leq$ on $S$ such that

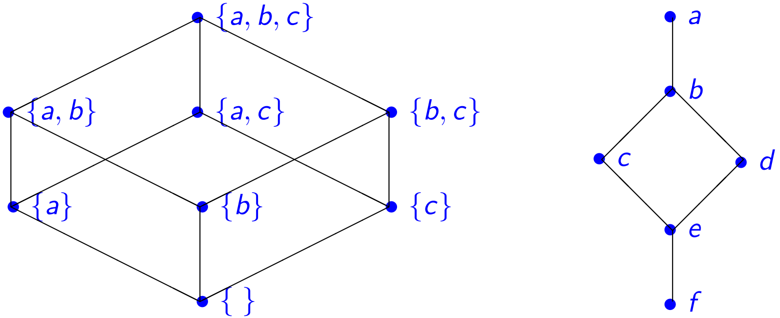
- **[Refl]** $s \leq s$ for all $s \in S$
- **[Antisym]** $s \leq t$ and $t \leq s$ impilies $s = t$, for all $s,\ t \in S$
- **[Trans]** $s \leq t$ and $t \leq u$ impilies $s \leq u$, for all $s,\ t,\ \in S$



# Upper Bounds and Complete Latices

- In a partial order $(S, \leq)$, given $X \subseteq S$, $y$ is an upper bound for $X$ if for all $x \in X$ we have $x \leq y$.
- $y$ is a least upper bound of $X$, $y$ is an upper bound of $X$ and whenever $z$ is an upper bound of $X$, $y \leq z$.
- **Note:** Least upper bounds are unique.
- A complete lattice is a partial order $(L, \leq)$ such that for all $X \subseteq S$ there exists a (unique) least upper bound.
- Write $\text{lub}(X)$ or $\bigvee X$ for the least upper bound of $X$.
- Write $x \vee y$ for $\bigvee\{x, y\}$
- **Note:** $x \vee y = x \Longleftrightarrow y \leq x$
- **Note:** Given a set $S$, $(\mathcal{P}(S), \subseteq)$ is a complete lattice.
- Write $\bot = \bigvee\{\ \}$ and $\top = \bigvee S$

# Example Complete Lattices



# Partial Orders, Functions, and Complete Lattices

- Let $X$ be an arbitrary set and $A$ and $B$ be partial orders.
- A function $f : A \to B$ is order-preserving if, for all $x, y \in A$ with $x \leq y$ we have $f(x) \leq f(y)$
- Function $f, g : X \to A$ may be ordered by pointwise comparrison:
  - Write $f \leq_{fun} g$ to mean that for all $x \in X$ we have $f(x) \leq g(x)$
  - Will leave off the subcript in general
- **Fact:** $(\{f \mid f : X \to B\}, \leq_{fun})$ is a partial order.
- **Fact:** $(\{f \mid f : X \to B\}, \leq_{fun})$ is a complete lattice if $B$ is.
- **Fact:** $(\{f \mid f : A \to B,\ f \text{ order-preserving}\} \leq_{fun})$ is a complete lattice if $B$ is.

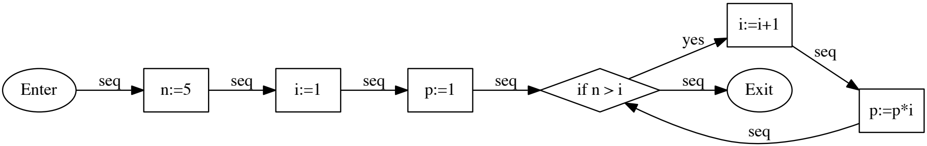
# Control-Flow Graphs

A Control-Flow Graph (for a SIMPL-like language) is a tuple $(N, l, K, E)$ where

- $N$ is a finite set of nodes
- $l : N \to \{\mathrm{Entry}, \mathrm{Exit}, \mathrm{i{:=}e}, \mathrm{if\ b}, \}$
- $K = \{\mathrm{yes}, \mathrm{seq}\}$
- $E \subseteq N \times K \times N$ such that
  - for all $m$, $n$, $n' \in N$ and $k \in K$, if $(m, k, n) \in E$ and $(m, k, n') \in E$ then $n = n'$
  - if $m \in N$ and $l(m) = \mathrm{Exit}$ then $|\{n \,|\, \exists k \in K.\,(m, k, n) \in E\}| = 0$
  - if $m \in N$ and $l(m) = \mathrm{Entry}$ or $l(m) = i := e$ for some identifier $i$ and expression $e$, and $(m, k, n) \in E$ then $k = \mathrm{seq}$
  - if $m \in N$ and $l(m) = \mathrm{if}\ b$ for some boolean expression $b$, then $|\{n \,|\, \exists k \in K.\,(m, k, n) \in E\}| = 2$

# Example

n:=5; i:=1; p:=1; while n>i do i:=i+1; p:=p*i od



# Abstract Interpretation

- Let $(N, I, K, E)$ be a control flow graph.
- An abstract interpretation of control flow graphs is a pair $(A, \mathcal{I})$ where
  - $A$ is a complete latice and
  - $\mathcal{I} : ((E \to A) \times E) \to A$ (think *next state information vector*)
  - for all $f, g \in (E \to A)$, for all $e \in E$, if $f \leq g$ then $\mathcal{I}(f, e) \leq \mathcal{I}(g, e)$

# Abstract Semantics

- Can define $\overline{\mathcal{I}} : (E \to A) \to (E \to A)$ by $\overline{\mathcal{I}}(f)(e) = \mathcal{I}(f, e)$
- **Fact:** $\overline{\mathcal{I}}$ is order-preserving
- **Tarski's Fixed-Point Theorem:** If $A$ is a complete lattice and $f : A \to A$ is order-preserving, then $f$ has both a least and a greatest fixed-point (may or may not be the same).
- **Fact:** There exist $c : E \to A$ such that $\overline{\mathcal{I}}(c) = c$, and that $c$ is the least such.
- Write $\mu\overline{\mathcal{I}}$ for the least fixed point of $\overline{\mathcal{I}}$
- $\mu\overline{\mathcal{I}}$ is the abstract semantics of $(N, I, K, E)$ with respect to $(A, \mathcal{I})$.

# Domain for Standard Interpretation

- Given $(N, I, K, E)$ a control flow graph with labels using variables from $Var$
- Let $Val = values \cup \{\top, \bot\}$, the extended set of values, ordered as before
  - $Val$ is a complete lattice.
- Let $Env = \{\rho \,|\, \rho : Var \to Val\}$
  - $Env$ is a complete lattice
  - An env used to be a partial function; now map undefined to $\bot$
  - $val : (Exp \times Env) \to Val$
  - Will assume $\{\text{true}, \text{false}\} \subseteq \text{values}$
  - $bval : (BExp \times Env) \to \{\text{true}, \text{false}\} \cup \{\top, \bot\} \subseteq Val$
- Let $States = (E \cup \{\top, \bot\}) \times Env$
  - $States$ is a complete lattice assuming the order $((e, \rho) \leq (e', \rho')) \equiv ((e \leq e') \wedge (\rho \leq \rho'))$.

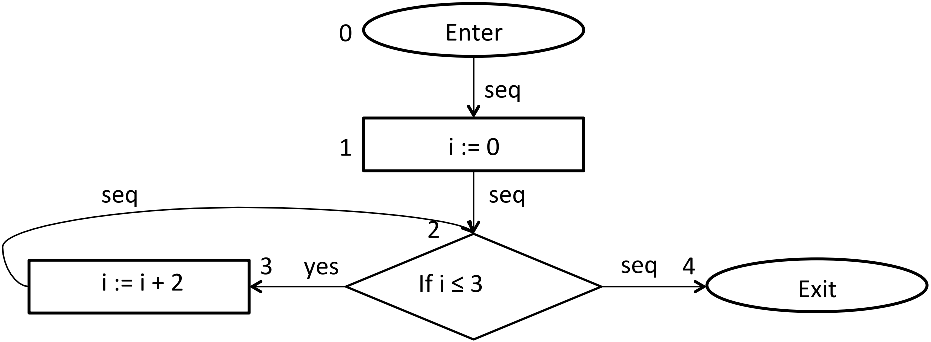
# Transitions in Control Flow Graphs

- next_state : $States \rightarrow States$
- $\text{next\_state}(\top, \rho) = (\top, \rho)$; $\text{next\_state}(\bot, \rho) = (\bot, \rho)$
- $\text{next\_state}((m, k, n), \rho)$ defined by cases on $l(n)$:
  - $l(n) \neq \text{Enter}$
  - $l(n) = \text{Exit} \Rightarrow \text{next\_state}((m, k, n), \rho) = ((m, k, n), \rho)$
  - $l(n) = (i := e)$, then $n$ has unique successor node $p$, $(n, \text{suc}, p) \in E$.
    - $\text{next\_state}((m, k, n), \rho) = ((n, \text{suc}, p), \rho[i \mapsto val(e, \rho)])$
  - $l(n) = (\mathit{if}\ \ b)$, then $n$ has two out arcs: $(n, \text{yes}, p)$ and $(n, \text{seq}, q)$
    - if $bval(b, \rho) = \bot$ then $\text{next\_state}((m, k, n), \rho) = (\bot, \rho)$
    - if $bval(b, \rho) = \top$ then $\text{next\_state}((m, k, n), \rho) = (\top, \rho)$
    - $bval(b, \rho) = \text{true}$ then $\text{next\_state}((m, k, n), \rho) = ((n, \text{yes}, p), \rho)$
    - $bval(b, \rho) = \text{false}$ then $\text{next\_state}((m, k, n), \rho) = ((n, \text{suc}, q), \rho)$
- next_state is transition semantics for control flow graphs

# Example

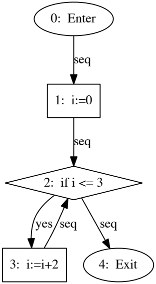
Consider the following control flow graph $(N, I, K, E)$ where:

- $Var = \{i\}, \quad values = \mathbb{Z}$
- $N = \{0, 1, 2, 3, 4, 5, 6\}$
- $I(0) = \text{Enter}$, $I(1) = \texttt{i:=0}$, $I(2) = \texttt{if 1} \leq \texttt{3}$, $I(3) = \texttt{i:=i+2}$, $I(4) = \text{Exit}$
- $K = \{\text{yes}, \text{seq}\}$
- $E = \left\{ \begin{array}{ll} (0, \text{seq}, 1), & (1, \text{seq}, 2), \\ (2, \text{yes}, 3), & (2, \text{seq}, 4), \\ (3, \text{seq}, 2) & \end{array} \right\}$


0
Enter
seq
1
i := 0
seq
seq
2
If i ≤ 3
3
yes
i := i + 2
seq
4
Exit

# Example: next_state

- $\text{next\_state}((0, \text{seq}, 1), \{i \mapsto \bot\}) = ((1, \text{seq}, 2), \{i \mapsto 0\})$
- $\text{next\_state}((1, \text{seq}, 2), \{i \mapsto 0\}) = ((2, \text{yes}, 3), \{i \mapsto 0\})$
- $\text{next\_state}((2, \text{yes}, 3), \{i \mapsto 0\}) =$
  $((3, \text{seq}, 2), \{i \mapsto 0\}[i \mapsto 0 + 2]) = ((3, \text{seq}, 2), \{i \mapsto 2\})$
- Since $\{i \mapsto 2\}(i) = 2 \leq 3$
  $\text{next\_state}((3, \text{seq}, 2), \{i \mapsto 2\}) = ((2, \text{yes}, 3), \{i \mapsto 2\})$
- $\text{next\_state}((2, \text{yes}, 3), \{i \mapsto 2\}) =$
  $((3, \text{seq}, 2), \{i \mapsto 2\}[i \mapsto 2 + 2]) = ((3, \text{seq}, 2), \{i \mapsto 4\})$
- Since $\{i \mapsto 4\}(i) = 4 \not\leq 3$
  $\text{next\_state}((3, \text{seq}, 2), \{i \mapsto 4\}) = ((2, \text{seq}, 4), \{i \mapsto 4\})$



# Standard Interpretation and Semantics

- Let $Interp(\theta, (m, k, n))$ be the lifting of next_state to sets of environments (contexts)
  - Note: $I(m) \neq \text{Exit}$
  - $I(m) = \text{Enter} \Rightarrow Interp(\theta, (m, k, n)) = \{\{v \mapsto \bot | v \in Var\}\} = \{\lambda v. \bot\}$
  - $I(m) \neq \text{Enter} \Rightarrow$
    $Interp(\theta, (m, k, n)) =$
    $\{\rho \mid \exists m', k', \rho' \mid (m', k', m) \in E \wedge$
    $\rho' \in \theta((m', k', m)) \wedge$
    $\text{next\_state}((m', k', m), \rho') = ((m, k, n), \rho)\}$
- If $\theta$ tells all the environments we might come into our edge with, $Interp(\theta, (m, k, n))$ tells us the set of environemts we may leave with

# Standard Interpretation and Semantics

- Let $\mathit{Contexts} = \mathcal{P}(\mathit{Env})$
  - $\mathit{Contexts}$ is a complete lattice
  - A context corresponds to a formula in prediacte logic over the program variables
- If for all $e \in E$ we have $\theta(e) \subseteq \phi(e)$, then for all $e' \in E$ we have $\mathit{Interp}(\theta, e') \subseteq \mathit{Interp}(\phi, e')$
- **Result:** $(\mathit{Contexts}, \mathit{Interp})$ is an abtract interpretation
- Recall: $\mathit{Interp} : ((E \rightarrow \mathit{Contexts}) \times E) \rightarrow \mathit{Contexts}$ so $\overline{\mathit{Interp}} : (E \rightarrow \mathit{Contexts}) \rightarrow (E \rightarrow \mathit{Contexts})$
- $\mu\, \overline{\mathit{Interp}}$ tells us the best knowledge we can know statically about our program

# Example: *Interp*

Let $\theta$ map edges to sets of environments. *Interp* will tell us the set of environments next_state will associate with each edge assuming $\theta$ gives a set of (possibly) possible environments for each predecessor edge:

- Since $Var = \{i\}$, $Interp(\theta, (0, \text{seq}, 1)) = \{\{i \mapsto \bot\}\}$
- If $\theta(e) = \{\ \}$ then $Interp(\theta, e) = \{\ \}$, so assume $\theta(e) \neq \{\ \}$
- $Interp(\theta, (1, \text{seq}, 2))$
  $= \{\rho \mid \exists \rho' \in \theta((0, \text{seq}, 1)) \,|\, \rho = \rho'[i \mapsto 0]\} = \{\{i \mapsto 0\}\}$
- $Interp(\theta, (2, \text{yes}, 3)) = \{\rho \in \theta(1, \text{seq}, 2) \cup \theta(3, \text{seq}, 2) \mid \rho(i) \leq 3\}$
- $Interp(\theta, (3, \text{seq}, 2)) = \{\rho \mid \exists \rho' \in \theta(2, \text{yes}, 3) \,|\, \rho = \rho`[i \mapsto \rho'(i) + 2]\}$
- $Interp(\theta, (2, \text{no}, 4)) = \{\rho \in \theta(3, \text{seq}, 2) \mid \rho(i) > 3\}$

- $\overline{Interp}(\theta)(e) = Interp(\theta, e)$
- $\overline{Interp}^0(\theta)(e) = \{\ \}$ $\quad \overline{Interp}^{n+1}(\theta)(e) = \overline{Interp}(\overline{Interp}^n(\theta))(e)$

# Example: $\mu\overline{Interp}$

- $\mu\overline{Inter} : E \to Contexts = \mathcal{P}(Env)$
- Start with minimal $\theta_0$ assigning no environments to any edge: $\theta_0(e) = \{\,\}$
- $\mu\overline{Interp}(e) = \bigcup_{n\in\mathbb{N}} \overline{Interp}^n(e)$
- $\mu\overline{Interp}(0, \mathrm{seq}, 1) = \{ \qquad \}$
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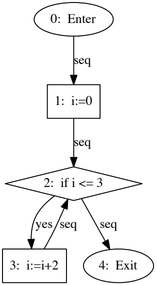

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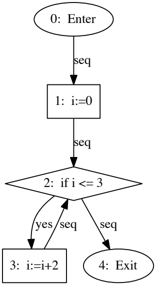

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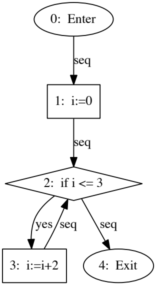

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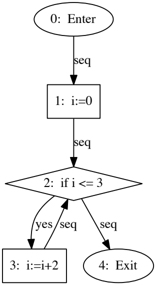

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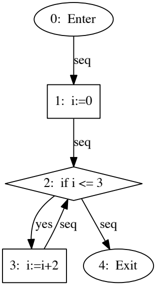

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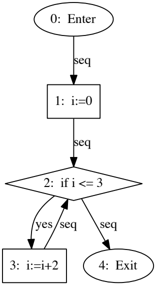

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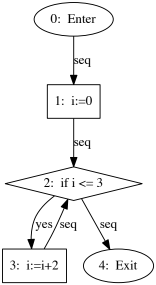

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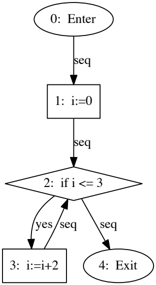

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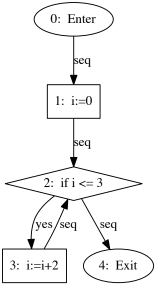

# Soundness of Abstract Semantics

**Fact:** An abstract interpretation $(A, \mathcal{I})$ is sound (or consistent) with respect to $(Env, Interp)$ if and only if there exist $\alpha,\ \beta$ such that

- $\alpha : Contexts \rightarrow A$, $\beta : A \rightarrow Contexts$
- $\alpha,\ \beta$ order preserving
- For all $a \in A$ have $\alpha(\beta(a)) = a$
- For all $S \in Contex$, have $S \subseteq \beta(\alpha(S))$
- For all $e \in E$, $\alpha(\mu\overline{Interp}(e)) = \mu\overline{\mathcal{I}}(e)$
  - The abtract interpretation gives us more possibilities, is less precise