# CS477 Formal Software Development Methods

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## Example: Traffic Light

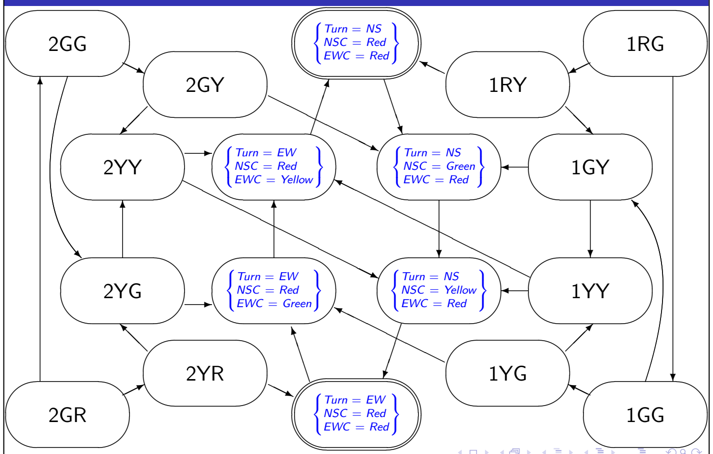
$V = \{Turn, NSC, EWC\}$, $F = \{NS, EW, Red, Yellow, Green\}$ (all arity 0), $R = \{=\}$

| | |
|---|---|
| $NSG$ | $Turn = NS \wedge NSC = Red \rightarrow NSC := Green$ |
| $NSY$ | $NSC = Green \rightarrow NSC := Yellow$ |
| $NSR$ | $NSC = Yellow \rightarrow (Turn, NSC) := (EW, Red)$ |
| $EWG$ | $Turn = EW \wedge EWC = Red \rightarrow EWC := Green$ |
| $EWY$ | $EWC = Green \rightarrow EWC := Yellow$ |
| $EWR$ | $EWC = Yellow \rightarrow (Turn, EWC) := (NS, Red)$ |

$init = (NSC = Red \wedge EWC = Red \wedge (Turn = NS \vee Turn = EW)$



## Example: Traffic Lights






## Examples (cont)

- LTS for traffic light has $3 \times 3 \times 2 = 18$ possible well typed states
  - Is is possible to reach a state where $NSC \neq Red \wedge EWC \neq Red$ from an initial state?
  - If so, what sequence of actions allows this?
  - Do all the immediate predecessors of a state where $NSC = Green \vee EWC = Green$ satisfy $NSC = Red \wedge EWC = Red$?
  - If not, are any of those offend states reachable from and initial state, and if so, how?
- LTS for Mutual Exclusion has $6 \times 6 \times 2 \times 2 = 144$ posible well-tped states.
  - Is is possible to reach a state where $pc1 = m5 \wedge pc2 = n5$?
- How can we state these questions rigorously, formally?
- Can we find an algorihm to answer these types of questions?



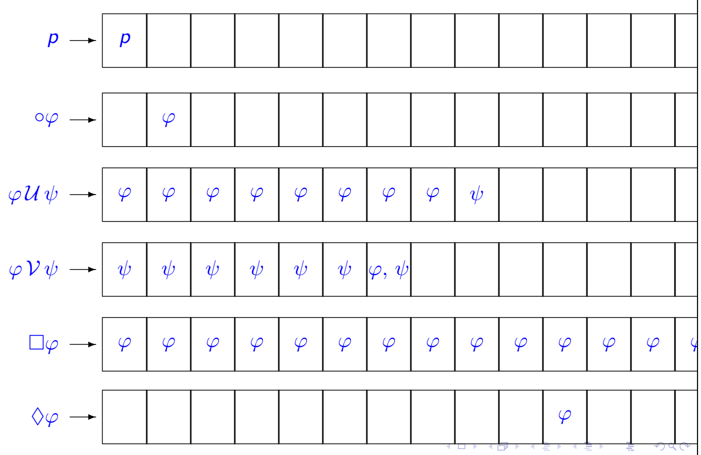
## Linear Temporal Logic - Syntax

$$\begin{aligned}\varphi \; ::= \; & p \mid (\varphi) \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi' \\ \mid \; & \circ\varphi \mid \varphi\,\mathcal{U}\,\varphi' \mid \varphi\,\mathcal{V}\,\varphi' \mid \Box\varphi \mid \Diamond\varphi\end{aligned}$$

- $p$ – a propostion over state variables
- $\circ\varphi$ – "next"
- $\varphi\mathcal{U}\varphi'$ – "until"
- $\varphi\mathcal{V}\varphi'$ – "releases"
- $\Box\varphi$ – "box", "always", "forever"
- $\Diamond\varphi$ – "diamond", "eventually", "sometime"



## LTL Semantics: The Idea



## Formal LTL Semantics

Given:

- $\mathcal{G} = (V, F, af, R, ar)$ signature expressing state propositions
- $Q$ set of states,
- $\mathcal{M}$ modeling function over $Q$ and $\mathcal{G}$: $\mathcal{M}(q,p)$ is true iff $q$ models $p$. Write $q \models p$.
- $\sigma = q_0 q_1 \dots q_n \dots$ infinite sequence of state from $Q$.
- $\sigma^i = q_i q_{i+1} \dots q_n \dots$ the $i^{th}$ tail of $\sigma$

Say $\sigma$ models LTL formula $\varphi$, write $\sigma \models \varphi$ as follows:

- $\sigma \models p$ iff $q_0 \models p$
- $\sigma \models \neg\varphi$ iff $\sigma \not\models \varphi$
- $\sigma \models \varphi \wedge \psi$ iff $\sigma \models \varphi$ and $\sigma \models \psi$.
- $\sigma \models \varphi \vee \psi$ iff $\sigma \models \varphi$ or $\sigma \models \psi$.



## Formal LTL Semantics

- $\sigma \models \circ\varphi$ iff $\sigma^1 \models \varphi$
- $\sigma \models \varphi \mathcal{U} \psi$ iff for some $k$, $\sigma^k \models \psi$ and for all $i < k$, $\sigma^i \models \varphi$
- $\sigma \models \varphi \mathcal{V} \psi$ iff for some $k$, $\sigma^k \models \varphi$ and for all $i \leq k$, $\sigma^i \models \psi$, or for all $i$, $\sigma^i \models \psi$.
- $\sigma \models \Box\varphi$ if for all $i$, $\sigma^i \models \psi$
- $\sigma \models \Diamond\varphi$ if for some $i$, $\sigma^i \models \psi$



## Some Common Combinations

- $\Box\Diamond p$ "$p$ will hold infinitely often"
- $\Diamond\Box p$ "$p$ will continuously hold from some point on"
- $(\Box p) \Rightarrow (\Box q)$ "if $p$ happens infinitely often, then so does $q$



## Some Equivalences

- $\Box(\varphi \wedge \psi) = (\Box\varphi) \wedge (\Box\psi)$
- $\Diamond(\varphi \vee \psi) = (\Diamond\varphi) \vee (\Diamond\psi)$
- $\Box\varphi = \mathbf{F}\,\mathcal{V}\,\varphi$
- $\Diamond\varphi = \mathbf{T}\,\mathcal{U}\,\varphi$
- $\varphi\,\mathcal{V}\,\psi = \neg((\neg\varphi)\,\mathcal{U}\,(\neg\psi))$
- $\varphi\,\mathcal{U}\,\psi = \neg((\neg\varphi)\,\mathcal{V}\,(\neg\psi))$
- $\neg(\Diamond\varphi) = \Box(\neg\varphi)$
- $\neg(\Box\varphi) = \Diamond(\neg\varphi)$



## Some More Eqivalences

- $\Box\varphi = \varphi \wedge \circ\Box\varphi$
- $\Diamond\varphi = \varphi \vee \circ\Diamond\varphi$
- $\varphi\,\mathcal{V}\,\psi = (\varphi \wedge \psi) \vee (\psi \wedge \circ(\varphi\,\mathcal{V}\,\psi))$
- $\varphi\,\mathcal{U}\,\psi = \psi \vee (\varphi \wedge \circ(\varphi\,\mathcal{V}\,\psi))$
- $\Box$, $\Diamond$, $\mathcal{U}$, $\mathcal{V}$ may all be understood recursively, by what they state about right now, and what they state about the future
- Caution: $\Box$ vs $\Diamond$, $\mathcal{U}$ vs $\mathcal{V}$ differ in there limit behavior



## Traffic Light Example

Basic Behavior:

- $\Box((NSC = Red) \vee (NSC = Green) \vee (NSC = Yellow))$
- $\Box((NSC = Red) \Rightarrow ((NSC \neq Green) \wedge (NSC \neq Yellow))$
- Similarly for *Green* and *Red*
- $\Box(((NCS = Red) \wedge \circ(NCS \neq Red)) \Rightarrow \circ(NCS = Green))$
- Same as $\Box((NCS = Red) \Rightarrow ((NCS = Red)\,\mathcal{U}\,(NCS = Green)))$
- $\Box(((NCS = Green) \wedge \circ(NCS \neq Green)) \Rightarrow \circ(NCS = Yellow))$
- $\Box(((NCS = Yellow) \wedge \circ(NCS \neq Yellow)) \Rightarrow \circ(NCS = Red))$
- Same for *EWC*

## Traffic Light Example

Basic Safety

- $\Box((NSC = Red) \vee (EWC = Red)$
- $\Box(\,((NSC = Red) \wedge (EWC = Red))\,\mathcal{V}$
  $((NSC \neq Green) \Rightarrow (\circ(NSC = Green))))$

Basic Liveness

- $(\Diamond(NSC = Red)) \wedge (\Diamond(NSC = Green)) \wedge (\Diamond(NSC = Yellow))$
- $(\Diamond(EWC = Red)) \wedge (\Diamond(EWC = Green)) \wedge (\Diamond(EWC = Yellow))$



## Proof System for LTL

- First step: View $\varphi\,\mathcal{V}\,\psi$ as moacro: $\varphi\,\mathcal{V}\,\psi = \neg((\neg\varphi)\,\mathcal{U}\,(\neg\psi))$
- Second Step: Extend all rules of Prop Logic to LTL
- Third Step: Add one more rule: $\dfrac{\Box\varphi}{\varphi}$ Gen
- Fourth Step: Add a collection of axioms (a sufficient set of 8 exists)
  - A1: $\Box\varphi \Leftrightarrow \neg(\Diamond(\neg\varphi))$
  - A2: $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$
  - A3: $\Box\varphi \Rightarrow (\varphi \wedge \circ\Box\varphi)$
  - A4: $\circ\neg\varphi \Leftrightarrow \neg\circ\varphi$
  - A5: $\circ(\varphi \Rightarrow \psi) \Rightarrow (\circ\varphi \Rightarrow \circ\psi)$
  - A6: $\Box(\varphi \Rightarrow \circ\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$
  - A7: $\varphi\,\mathcal{U}\,\psi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \circ(\varphi\,\mathcal{V}\,\psi)$
  - A8: $\varphi\,\mathcal{U}\,\psi \Rightarrow \Diamond\psi$
- Result: a sound and relatively complete proof system
- Can implement in Isabelle in much the same way as we did Hoare Logic



## Important Meta-Definitions

- $A$ is sound with respect to $B$ if things that are "true" according to $A$ are things that are "true" according to $B$.
- $A$ is complete with respect to $B$ if things that are "true" according to $B$ are things that are "true" according to $A$.
- $A$ is sound if things that are "true" according to $A$ are true.
- $A$ is complete everything that is true (that is in the scope of $A$) is "true" according to $A$.
- $A$ is relatively complete with repsect to $B$ if $A$ is complete when $B$ is. Think: $A$ proof system, $B$ mathematical model; or $A$ a proof system, $B$ a subsystem.



## Exercise: $\varphi \wedge \psi \Rightarrow \Box\varphi \wedge \Box$



## What is Model Checking?

Most generally Model Checking is

- an automated technique, that given
- a finite-state model $M$ of a system
- and a logical property $\varphi$,
- checks whether the property holds of model: $M \models \varphi$?



## Model Checking

- Model checkers usually give example of failure if $M \not\models \varphi$.
- This makes them useful for debugging.
- Problem: Can only handle finite models: unbounded or continuous data sets can't be directly handled
- Problem: Nnmber of states grows exponentially in the size of the system.
- Answer: Use abstract model of system
- Problem: Relationship of results on abstract model to real system?

## LTL Model Checking

- Model Checking Problem: Given model $\mathcal{M}$ amd logical property *varphi* of $\mathcal{M}$, does $\mathcal{M} \models \varphi$?
- Given transition system with states $Q$, transition relation $\delta$ and inital state state $I$, say $(Q, \delta, I) \models \varphi$ for LTL formula $\varphi$ if every run of $(Q, \delta, I)$, $\sigma$ satisfies $\sigma \models \varphi$.

**Theorem**

*The Model Checking Problem for finite transition systems and LTL formulae is decideable.*

- Treat states $q \in Q$ as letters in an alphabet.
- Language of $(Q, \delta, I)$, $\mathcal{L}(Q, \delta, I)$ (or L(Q) for short) is set of runs in $Q$
- Language of $\varphi$, $\mathcal{L}\varphi = \{\sigma | \sigma \models \varphi\}$
- Question: $\mathcal{L}(Q) \subseteq \mathcal{L}(\varphi)$?
- Same as: $\mathcal{L}(Q) \cap \mathcal{L}(\neg\varphi) = \emptyset$?