CS477 Formal Software Development Methods

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First Order Logic vs Propositional Logic

First Order Logic extends Propositional Logic with

- Non-boolean constants
- Variables
- Functions and relations (or predicates, more generally)
- Quantification of variables

Sample first order formula:

$$\forall x.\exists y.x < y \wedge y \leq x + 1$$

Reference: Peled, *Software Reliability Methods*, Chapter 3

Signatures

Start with signature:

$$\mathcal{G} = (V, F, af, R, ar)$$

- V a countably infinite set of *variables*
- F finite set of function symbols
- $af : F \to \mathbb{N}$ gives the *arity*, the number of arguments for each function Constant c is a function symbol of arity 0 ($af(c) = 0$)
- R finite set of relation symbols
- $ar : R \to \mathbb{N}$, the arity for each relation symbol
  - Assumes $= \in R$ and $ar(=) = 2$

Terms over Signature

Terms t are expressions built over a signature (V, F, af, R, ar)

$$
\begin{array}{rlll}
t ::= & v & & v \in V \\
| & f(t_1, \ldots, t_n) & & f \in F \text{ and } n = af(f)
\end{array}
$$

- **Example**: $add(1, abs(x))$ where $add,\ abs, 1 \in F;\ x \in V$
- For constant c write c instead of $c(\ )$
- Will write $s = t$ instead of $= (s, t)$
  - Similarly for other common infixes (*e.g.* $+$, $-$, $*$, $<$, $\leq, \ldots$)

Structures

Meaning of terms starts with a structure:

$$\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$$

where

- $\mathcal{G} = (V, F, af, R, ar)$ a signature,
- $\mathcal{D}$ and *domain* on interpretation
- $\mathcal{F}$ set of functions over $\mathcal{D}$; $\quad \mathcal{F} \subseteq \bigcup_{n \geq 0} \mathcal{D}^n \to \mathcal{D}$
  - **Note:** $\mathcal{F}$ can contain elements of $\mathcal{D}$ since $\mathcal{D} = (\mathcal{D}^0 \to \mathcal{D})$
- $\phi : F \to \mathcal{F}$ where if $\phi(f) \in (\mathcal{D}^n \to \mathcal{D})$ then $n = af(f)$
- $\mathcal{R}$ set of relations over $\mathcal{D}$; $\quad \mathcal{R} \subseteq \bigcup_{n \geq 1} \mathcal{P}(\mathcal{D}^n)$
- $\rho : R \to \mathcal{R}$ where if $\rho(r) \subseteq \mathcal{D}^n$ then $n = ar(r)$

Assignments

V set of variables, $\mathcal{D}$ domain of interpretation
An assignment is a function $a : V \to \mathcal{D}$
Example:

$$V = \{w, x, y, z\}$$

$$a = \{w \mapsto 3.14, x \mapsto -2.75, y \mapsto 13.9, z \mapsto -25.3\}$$

- Assignment is a fixed association of values to variables; not "update-able"

Interpretation of Terms

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a : V \to \mathcal{D}$, the interpretation $\mathcal{T}_a$ of a term t is defined by structural induction on terms:

- $\mathcal{T}_a(v) = a(v)$ for $v \in V$
- $\mathcal{T}_a(f(t_1, \ldots, t_n)) = \phi(f)(\mathcal{T}_a(t_1), \ldots, \mathcal{T}_a(t_n))$

Example of Interpretation

- $V = \{w, x, y, z\}$, $\mathcal{D} = \mathbb{R}$
- $1, add, abs \in F$, constant 1, and functions (in $\mathcal{F}$) for addition and absolute value respectively
- $a = \{w \mapsto 3.14, x \mapsto -2.75, y \mapsto 13.9, z \mapsto -25.3\}$

$$\begin{aligned}
\mathcal{T}_a(add(1, abs(x))) &= (\mathcal{T}_a(1)) + (\mathcal{T}_a(abs(x))) \\
&= 1.0 + (\mathcal{T}_a(abs(x))) \\
&= 1.0 + |\mathcal{T}_a(x)| \\
&= 1.0 + |a(x)| \\
&= 1.0 + |-2.75| \\
&= 1.0 + 2.75 \\
&= 3.75
\end{aligned}$$

First-Order Formulae

First-order formulae built from terms using relations, logical connectives, quantifiers:

$$
\begin{array}{rll}
\mathit{form} ::= & \mathrm{true} \;\mid\; \mathrm{false} & \\
\mid & r(t_1, \ldots, t_n) & r \in R,\;\; t_i \text{ terms},\;\; n = ar(r) \\
\mid & (\mathit{form}) \mid \neg \mathit{form} & \\
\mid & \mathit{form} \wedge \mathit{form} & \\
\mid & \mathit{form} \vee \mathit{form} & \\
\mid & \mathit{form} \Rightarrow \mathit{form} & \\
\mid & \mathit{form} \Leftrightarrow \mathit{form} & \\
\mid & \forall v.\mathit{form} & \\
\mid & \exists v.\mathit{form} &
\end{array}
$$

Note: Scope of quantifiers as far to right as possible

$$
\begin{array}{rrl}
\forall x.(x > y) \wedge (2 > x) & \text{same as} & \forall x.((x > y) \wedge (2 > x)) \\
 & \text{not same as} & (\forall x.(x > y)) \wedge (2 > x)
\end{array}
$$

Subformulae

- A subformula of formula ψ is a formula that occurs in ψ
  - More rigorous definition by structural induction on formulae
  - ψ subformula of ψ
  - Use proper subformula to exclude ψ
- Write $\bigwedge_{i=1,\ldots,n} \psi_i$ for $\psi_1 \wedge \ldots \wedge \psi_n$
  - ψ_i called a conjunct
- Write $\bigvee_{i=1,\ldots,n} \psi_i$ for $\psi_1 \vee \ldots \vee \psi_n$
  - ψ_i called a disjunct

Free Variables: Terms

Informally: free variables of a expression are variables that have an occurrence in an expression that is not bound. Written $fv(e)$ for expression e

Free variables of terms defined by structural induction over terms; written

- $fv(x) = \{x\}$
- $fv(f(t_1, \ldots, t_n)) = \bigcup_{i=1,\ldots,n} fv(t_i)$

Note:

- Free variables of term just variables occurring in term; no bound variables
- No free variables in constants
- **Example**: $fv(add(1, abs(x))) = \{x\}$

Free Variables: Formulae

Defined by structural induction on formulae; uses *fv* on terms

- $fv(\mathrm{true}) = fv(\mathrm{false}) = \{\ \}$
- $fv(r(t_1, \ldots, t_n)) = \bigcup_{i=1,\ldots,n} fv(t_i)$
- $fv(\psi_1 \wedge \psi_2) = fv(\psi_1 \vee \psi_2) = fv(\psi_1 \Rightarrow \psi_2) = fv(\psi_1 \Leftrightarrow \psi_2) = (fv(\psi_1) \cup fv(\psi_2))$
- $fv(\forall v.\, \psi) = fv(\exists v.\, \psi) = (fv(\psi) \setminus \{v\})$

Variable occurrence at quantifier are binding occurrence

Occurrence that is not free and not binding is a bound occurrence

Example: $fv(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y))) = \{x, z\}$

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a : V \to \mathcal{D}$, the interpretation $\mathcal{M}_a$ of a formula ψ assigning a value in $\{\mathbf{T}, \mathbf{F}\}$ is defined by structural induction on formulae:

Interpretation of Formulae

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- $\mathcal{M}_a(\mathrm{true}) = \mathbf{T}$ $\qquad\qquad \mathcal{M}_a(\mathrm{false}) = \mathbf{F}$

Interpretation of Formulae

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- $\mathcal{M}_a(\mathrm{true}) = \mathbf{T}$ $\qquad$ $\mathcal{M}_a(\mathrm{false}) = \mathbf{F}$
- $\mathcal{M}_a(r(t_1, \ldots, t_n)) = \rho(r)(\mathcal{T}_a(t_1), \ldots, \mathcal{T}_a(t_n))$

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

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- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

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- $\mathcal{M}_a(\mathrm{true}) = \mathbf{T}$ $\qquad$ $\mathcal{M}_a(\mathrm{false}) = \mathbf{F}$
- $\mathcal{M}_a(r(t_1, \ldots, t_n)) = \rho(r)(\mathcal{T}_a(t_1), \ldots, \mathcal{T}_a(t_n))$
- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$
- $\mathcal{M}_a(\neg\psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg\psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a : V \rightarrow \mathcal{D}$, the interpretation $\mathcal{M}_a$ of a formula ψ assigning a value in $\{\mathbf{T}, \mathbf{F}\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\mathrm{true}) = \mathbf{T}$ $\quad\quad$ $\mathcal{M}_a(\mathrm{false}) = \mathbf{F}$
- $\mathcal{M}_a(r(t_1, \ldots, t_n)) = \rho(r)(\mathcal{T}_a(t_1), \ldots, \mathcal{T}_a(t_n))$
- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$
- $\mathcal{M}_a(\neg\psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg\psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ and $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a : V \to \mathcal{D}$, the interpretation $\mathcal{M}_a$ of a formula ψ assigning a value in $\{\mathbf{T}, \mathbf{F}\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\mathrm{true}) = \mathbf{T}$ $\qquad$ $\mathcal{M}_a(\mathrm{false}) = \mathbf{F}$
- $\mathcal{M}_a(r(t_1, \ldots, t_n)) = \rho(r)(\mathcal{T}_a(t_1), \ldots, \mathcal{T}_a(t_n))$
- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$
- $\mathcal{M}_a(\neg\psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg\psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ and $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{F}$ otherwise
- $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ or $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

For given assignment $a : V \to \mathcal{D}$, the interpretation $\mathcal{M}_a$ of a formula ψ assigning a value in $\{\mathbf{T}, \mathbf{F}\}$ is defined by structural induction on formulae:

- $\mathcal{M}_a(\text{true}) = \mathbf{T}$ $\qquad\qquad$ $\mathcal{M}_a(\text{false}) = \mathbf{F}$
- $\mathcal{M}_a(r(t_1, \ldots, t_n)) = \rho(r)(\mathcal{T}_a(t_1), \ldots, \mathcal{T}_a(t_n))$
- $\mathcal{M}_a((\psi)) = \mathcal{M}_a(\psi)$
- $\mathcal{M}_a(\neg\psi) = \mathbf{T}$ if $\mathcal{M}_a(\psi) = \mathbf{F}$ and $\mathcal{M}_a(\neg\psi) = \mathbf{F}$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ and $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \wedge \psi_2) = \mathbf{F}$ otherwise
- $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{T}$ or $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \vee \psi_2) = \mathbf{F}$ otherwise
- $\mathcal{M}_a(\psi_1 \Rightarrow \psi_2) = \mathbf{T}$ if $\mathcal{M}_a(\psi_1) = \mathbf{F}$ or $\mathcal{M}_a(\psi_2) = \mathbf{T}$, and $\mathcal{M}_a(\psi_1 \Rightarrow \psi_2) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

Let

$$a + [v \mapsto d]\ (w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

Let

$$(a + [v \mapsto d])(w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

Let

$$a + [v \mapsto d]\ (w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

- $\mathcal{M}_a(\forall v.\psi) = \mathbf{T}$ if for every $d \in \mathcal{D}$ we have $\mathcal{M}_{a+[v\mapsto d]}(\psi) = \mathbf{T}$, and $\mathcal{M}_a(\forall v.\psi) = \mathbf{F}$ otherwise

Interpretation of Formulae

Fix structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

Let

$$a + [v \mapsto d]\ (w) = \begin{cases} d & \text{if } w = v \\ a(w) & \text{if } w \neq v \end{cases}$$

- $\mathcal{M}_a(\forall v.\psi) = \mathbf{T}$ if for every $d \in \mathcal{D}$ we have $\mathcal{M}_{a+[v \mapsto d]}(\psi) = \mathbf{T}$, and $\mathcal{M}_a(\forall v.\psi) = \mathbf{F}$ otherwise
- $\mathcal{M}_a(\exists v.\psi) = \mathbf{T}$ if there exists $d \in \mathcal{D}$ such that $\mathcal{M}_{a+[v \mapsto d]}(\psi) = \mathbf{T}$, and $\mathcal{M}_a(\forall v.\psi) = \mathbf{F}$ otherwise

Modeling First-order Formulae

Given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$ where $\mathcal{G} = (V, F, af, R, ar)$

- $(\mathcal{S}, \mathcal{M})$ model for first-order language over signature $\mathcal{G}$
- Truth of formulae in language over signature $\mathcal{G}$ depends on structure $\mathcal{S}$
- Assignment a models ψ, or a satisfies ψ, or $a \models^{\mathcal{S}} \psi$ if $\mathcal{M}_a(\psi) = \mathbf{T}$
- ψ is valid for $\mathcal{S}$ if $a \models^{\mathcal{S}} \psi$ for some a.
- $\mathcal{S}$ is a model of ψ, written $\models^{\mathcal{S}} \psi$ if every assignment for $\mathcal{S}$ satisfies ψ.
- ψ is valid, or a tautology if ψ valid for every mode. Write $\models \psi$
- ψ_1 logically equivalent to ψ_2 if for all structures $\mathcal{S}$ and assignments a, $a \models^{\mathcal{S}} \psi_1$ iff $a \models^{\mathcal{S}} \psi_2$

Examples

- Assignment $\{x \mapsto 0\}$ satisfies $\exists y. x < y$ valid in interval $[0,\ 1]$; assignment $\{x \mapsto 1\}$ doesn't
- $\forall x. \exists y. x < y$ valid in $\mathbb{N}$ and $\mathbb{R}$, but not interval $[0,\ 1]$
- $(\exists x. \forall y. (y \leq x)) \Rightarrow (\forall y. \exists x. (y \leq x))$ tautology
    - Why?

Sample Tautologies

All instances of propositional tautologies

Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \leq x)) \Rightarrow (\forall y. \exists x. (y \leq x))$$

Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x.\forall y.(y \leq x)) \Rightarrow (\forall y.\exists x.(y \leq x))$$

$$\models ((\forall x.\forall y.\psi) \Leftrightarrow (\forall y.\forall x.\psi))$$

Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x.\forall y.(y \leq x)) \Rightarrow (\forall y.\exists x.(y \leq x))$$

$$\models ((\forall x.\forall y.\psi) \Leftrightarrow (\forall y.\forall x.\psi))$$

$$\models ((\forall x.\psi) \Rightarrow (\exists x.\psi))$$

Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x. \forall y. (y \leq x)) \Rightarrow (\forall y. \exists x. (y \leq x))$$

$$\models ((\forall x. \forall y. \psi) \Leftrightarrow (\forall y. \forall x. \psi))$$

$$\models ((\forall x. \psi) \Rightarrow (\exists x. \psi))$$

$$\models (\forall x. \psi_1 \wedge \psi_2) \Leftrightarrow ((\forall x. \psi_1) \wedge (\forall x. \psi_2))$$

Sample Tautologies

All instances of propositional tautologies

$$\models (\exists x.\forall y.(y \leq x)) \Rightarrow (\forall y.\exists x.(y \leq x))$$

$$\models ((\forall x.\forall y.\psi) \Leftrightarrow (\forall y.\forall x.\psi))$$

$$\models ((\forall x.\psi) \Rightarrow (\exists x.\psi))$$

$$\models (\forall x.\psi_1 \wedge \psi_2) \Leftrightarrow ((\forall x.\psi_1) \wedge (\forall x.\psi_2))$$

$$(\exists x.\psi_1 \wedge \psi_2) \Rightarrow ((\exists x.\psi_1) \wedge (\exists x.\psi_2))$$

Free Variables, Assignments and Interpretation

Theorem

Assume given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$*, term* t *over* $\mathcal{G}$*, and* a *and* b *assignments. If for every* $x \in fv(t)$ *we have* $a(x) = b(x)$ *then* $\mathcal{T}_a(t) = cT_b(a)$*.*

Theorem

Assume given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$*, formula* ψ *over* $\mathcal{G}$*, and* a *and* b *assignments. If for every* $x \in fv(\psi)$ *we have* $a(x) = b(x)$ *then* $\mathcal{M}_a(\psi) = \mathcal{M}_b(\psi)$*.*

Syntactic Substitution versus Assignment Update

- When interpreting universal quantification ($\forall x.\, \psi$), wanted to check interpretation of every instance of ψ where v was replaced by element of semantic domain $\mathcal{D}$
- How: semantically - interpret ψ with assignment updated by $v \mapsto d$ for every $d \in \mathcal{D}$
- Syntactically?
- Answer: substitution

Substitution in Terms

- Substitution of term t for variable x in term s (written $s[t/x]$) gotten by replacing every instance of x in s by t
  - x called redex; t called residue
- Yields *instance* of s

Formally defined by structural induction on terms:

- $x[t/x] = t$
- $y[t/x] = y$ for variable y where $y \neq x$
- $f(t_1, \ldots, t_n)[t/x] = f(t_1[t/x], \ldots, t_n[t/x])$

Example: $(add(1, abs(x)))[add(x, y)/x] = add(1, abs(add(x, y)))$

Substitution in Formulae: Problems

- Want to define by structural induction, similar to terms
- Quantifiers must be handled with care
  - Substitution only replaces free occurrences of variable
    Example:

$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))[x + 2/z] =$$
$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (x + 2 \geq y)))$$

  - Need to avoid *free variable capture*
    Example Problem:

$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))[x + y/z] \neq$$
$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (x + y \geq y)))$$

Theorem

Assume given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$*, variable* x*, terms* s *and* t *over* $\mathcal{G}$*, and* a *assignment. Let* $b = a[x \mapsto \mathcal{T}_a(t)]$*. Then* $\mathcal{T}_a(s[t/x]) = \mathcal{T}_b(s)$*.*

Substitution in Formulae: Two Approaches

- When quantifier would capture free variable of redex, can't substitute in formula as is
- Solution 1: Make substitution partial function – undefined in this case
- Solution 2: Define equivalence relation based on renaming bound variables; define substitution on equivalence classes
- Will take Solution 1 here
- Still need definition of equivalence up to renaming bound variables

Substitution in Formulae

- Defined by structural induction; uses substitution in terms
- Read equations below as saying left is not defined if any expression on right not defined
- $\mathrm{true}[t/x] = \mathrm{true} \qquad \mathrm{false}[t/x] = \mathrm{false}$
- $r(t_1, \ldots, t_n)[t/x] = r((t_1[t/x], \ldots, t_n[t/x]))$
- $(\psi)[t/x] = (\psi[t/x]) \qquad (\neg\psi)[t/x] = \neg(\psi[t/x])$
- $(\psi_1 \otimes \psi_2)[t/x] = (\psi_1[t/x]) \otimes (\psi_2[t/x])$ for $\otimes \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$
- $(\mathcal{Q}\, x.\, \psi)[t/x] = \mathcal{Q}\, x.\, \psi$ for $\mathcal{Q} \in \{\forall, \exists\}$
- $(\mathcal{Q}\, y.\, \psi)[t/x] = \mathcal{Q}\, y.\, (\psi[t/x])$ if $x \neq y$ and $y \notin fv(t)$ for $\mathcal{Q} \in \{\forall, \exists\}$
- $(\mathcal{Q}\, y.\, \psi)[t/x]$ not defined if $x \neq y$ and $y \in fv(t)$ for $\mathcal{Q} \in \{\forall, \exists\}$

Substitution in Formulae

Examples

$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))[x + y/z]$ not defined

$$\begin{array}{l}(x > 3 \wedge (\exists w.\ (\forall z.\ z \geq (w - x)) \vee (z \geq w)))[x + y/z] = \\ (x > 3 \wedge (\exists w.\ (\forall z.\ z \geq (w - x)) \vee ((x + y) \geq y)))\end{array}$$

Theorem

Assume given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, formula ψ over $\mathcal{G}$, and a assignment. If $\psi[t/x]$ defined, then $a \models^{\mathcal{S}} \psi[t/x]$ if and only if $a[x \mapsto \mathcal{T}_a(t)] \models^{\mathcal{S}} \psi$

Renaming by Swapping: Terms

Define the swapping of two variables in a term $t[x \leftrightarrow y]$ by structural induction on terms:

- $x[x \leftrightarrow y] = y$ and $y[x \leftrightarrow y] = x$
- $z[x \leftrightarrow y] = z$ for z a variable, $z \neq x$, $z \neq y$
- $f(t_1, \ldots, t_n)[x \leftrightarrow y] = f(t_1[x \leftrightarrow y], \ldots, t_n[x \leftrightarrow y])$

Examples:

$$\begin{aligned} add(1, abs(add(x, y)))[x \leftrightarrow y] &= add(1, abs(add(y, x))) \\ add(1, abs(add(x, y)))[x \leftrightarrow z] &= add(1, abs(add(z, y))) \end{aligned}$$

Renaming by Swapping: Terms

Theorem

Assume given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, *variables* x *and* y, *term* t *over* $\mathcal{G}$, *and* a *assignment. Let* $b = a[x \mapsto a(y)][y \mapsto a(x)]$. *Then* $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$

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Proof.

By structural induction on terms, suffices to show theorem for the case where t variable, and case $t = f(t_1, \ldots, t_n)$, assuming result for $t_1, \ldots, t_n$

- Case: t variable
  - Subcase: $t = x$. Then $\mathcal{T}_a(x[x \leftrightarrow y]) = \mathcal{T}_a(y) = a(y)$ and $\mathcal{T}_b(x) = b(x) = a[x \mapsto a(y)][y \mapsto a(x)](x) = a[x \mapsto \mathcal{T}_a(y)](x) = a(y)$ so $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$
  - Subcase: $t = y$. Then $\mathcal{T}_a(y[x \leftrightarrow y]) = \mathcal{T}_a(x) = a(x)$ and $\mathcal{T}_b(y) = b(y) = a[x \mapsto a(y)][y \mapsto a(x)](x) = a(x)$ so $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$
  - Subcase: $t = z$ variable, $z \neq x$ and $z \neq y$. Then $\mathcal{T}_a(z[x \leftrightarrow y]) = \mathcal{T}_a(z) = a(z)$ and $\mathcal{T}_b(z) = b(z) = a[x \mapsto a(y)][y \mapsto a(x)](z) = a[x \mapsto \mathcal{T}_a(y)](z) = a(z)$ so $\mathcal{T}_a(t[x \leftrightarrow y]) = \mathcal{T}_b(t)$

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Proof.

- Case: $t = f(t_1, \ldots, t_n)$. Assume $\mathcal{T}_a(t_i[x \leftrightarrow y]) = \mathcal{T}_b(t_i)$ for $i = 1, \ldots, n$. Then

$$
\begin{aligned}
\mathcal{T}_a(t[x \leftrightarrow y]) &= \mathcal{T}_a(f(t_1, \ldots, t_n)[x \leftrightarrow y]) \\
&= \mathcal{T}_a(f(t_1[x \leftrightarrow y], \ldots, t_n[x \leftrightarrow y])) \\
&= \phi(f)(\mathcal{T}_a(t_1[x \leftrightarrow y]), \ldots, \mathcal{T}_a(t_n[x \leftrightarrow y])) \\
&= \phi(f)(\mathcal{T}_b(t_1), \ldots, \mathcal{T}_b(t_n)) \\
&\quad \text{since } \mathcal{T}_a(t_i[x \leftrightarrow y]) = \mathcal{T}_b(t_i) \text{ for } i = 1, \ldots, n \\
&= \mathcal{T}_b(f(t_1, \ldots, t_n)) \\
&= \mathcal{T}_b(t) \quad \square
\end{aligned}
$$

Renaming by Swapping: Formulae

Define the swapping of two variables in a formula $\psi[x \leftrightarrow y]$ by structural induction, using swapping on terms:

- $\mathrm{true}[x \leftrightarrow y] = \mathrm{true} \qquad \mathrm{false}[x \leftrightarrow y] = \mathrm{false}$
- $r(t_1, \ldots, t_n)[x \leftrightarrow y] = r((t_1[x \leftrightarrow y], \ldots, t_n[x \leftrightarrow y]))$
- $(\psi)[x \leftrightarrow y] = (\psi[x \leftrightarrow y]) \qquad (\neg\psi)[x \leftrightarrow y] = \neg(\psi[x \leftrightarrow y])$
- $(\psi_1 \otimes \psi_2)[x \leftrightarrow y] = (\psi_1[x \leftrightarrow y]) \otimes (\psi_2[x \leftrightarrow y])$ for $\otimes \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$
- $(\mathcal{Q}\, x.\, \psi)[x \leftrightarrow y] = \mathcal{Q}\, y.\, (\psi[x \leftrightarrow y])$ for $\mathcal{Q} \in \{\forall, \exists\}$
- $(\mathcal{Q}\, y.\, \psi)[x \leftrightarrow y] = \mathcal{Q}\, y.\, (\psi[x \leftrightarrow y])$ for $\mathcal{Q} \in \{\forall, \exists\}$
- $(\mathcal{Q}\, z.\, \psi)[x \leftrightarrow y] = \mathcal{Q}\, z.\, (\psi[x \leftrightarrow y])$ for z a variable with $z \neq x$, $z \neq y$, and $\mathcal{Q} \in \{\forall, \exists\}$

Renaming by Swapping: Formulae

Examples

$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))[x \leftrightarrow y]$$
$$= (y > 3 \wedge (\exists x.\ (\forall z.\ z \geq (x - y)) \vee (z \geq x)))$$

$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))[y \leftrightarrow z]$$
$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))[y \leftrightarrow w]$$

Theorem

Assume given structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \phi, \mathcal{R}, \rho)$, variables x and y, formula ψ over $\mathcal{G}$, and a assignment. If $x \notin fv(t)$ and $y \notin fv(t)$ then $\psi[x \leftrightarrow y] \equiv \psi$

α-equivalence

- $\psi \overset{\alpha}{\equiv} \psi$
- If $\psi_1 \overset{\alpha}{\equiv} \psi_2$ then $\psi_2 \overset{\alpha}{\equiv} \psi$.
- It $\psi_1 \overset{\alpha}{\equiv} \psi_2$ and $\psi_2 \overset{\alpha}{\equiv} \psi_3$ then $\psi_1 \overset{\alpha}{\equiv} \psi_3$
- If $x \notin fv(\psi)$ and $y \notin fv(\psi)$ then $\psi \overset{\alpha}{\equiv} \psi[x \leftrightarrow y]$.
- If $\psi_i \overset{\alpha}{\equiv} \psi_i'$ for $i = 1, 2$ then
  - $(\psi_1) \overset{\alpha}{\equiv} (\psi_1') \qquad \neg\psi_1 \overset{\alpha}{\equiv} \neg\psi_1'$
  - $\psi_1 \otimes \psi_2 \overset{\alpha}{\equiv} \psi_1' \otimes \psi_2'$ for $\otimes \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$
  - $\mathcal{Q}\, z.\, \psi_1 \overset{\alpha}{\equiv} \mathcal{Q}\, z.\, \psi_1'$ for $\mathcal{Q} \in \{\forall, \exists\}$

α-equivalence: Example

$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))$$
$$\stackrel{\alpha}{\equiv} (x > 3 \wedge (\exists w.\ (\forall z.\ z \geq (w - x)) \vee (z \geq w)))$$

$$(x > 3 \wedge (\exists y.\ (\forall z.\ z \geq (y - x)) \vee (z \geq y)))$$
$$\stackrel{\alpha}{\equiv} (x > 3 \wedge (\exists w.\ (\forall y.\ y \geq (w - x)) \vee (z \geq w)))$$

Proof Rules

Natural Deduction rules:
All rules from Propositional Logic included

$$\frac{\Gamma \vdash \psi[t/x]}{\Gamma \vdash \exists x.psi} \; ExI \qquad \frac{\Gamma \vdash \exists x.psi \quad}{}$$

$$\frac{\Gamma \vdash \psi[y/x] \quad y \notin (fv(\psi) \setminus \{x\}) \cup \bigcup \psi' \in \Gamma fv(\psi')}{\Gamma \vdash \forall x.\psi} \; AllI$$