CS477 Formal Software Development Methods

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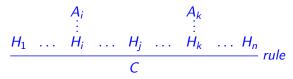http://courses.engr.illinois.edu/cs477

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Proofs in Propositional Logic

- Natural Deduction proofs are
  - trees with nodes that are inference rules
  - and a discharge function
- Inference rule has hypotheses and conclusion
- Conclusion: a single proposition
- Hypotheses: zero or more propositions, each possibly with hypotheses
  - Rule with no hypotheses called an axiom
- Inference rule graphically presents as

$$
\cfrac{H_1 \quad \ldots \quad \overset{\overset{A_i}{\vdots}}{H_i} \quad \ldots \quad H_j \quad \ldots \quad \overset{\overset{A_k}{\vdots}}{H_k} \quad \ldots \quad H_n}{C} \; rule
$$

- Discharge function: maps proof tree leaves with node (induction rule) introducing it
- Instance of inference rule above may discharge A_i and A_j in a proof using *rule*

Inference Rules

- Inference rules associated with connectives
- Two main kinds of (non-axiom) inference rules:
  - Introduction – says how to conclude proposition made from connective is true
    - Proposition below the line is instance of the connective
    - Hypotheses are (some of) connective components
  - Eliminations – says how to use a proposition made from connective to prove result
  - Conclusion: propositional atom (*i.e.* any conclusion can be instance)
  - Hypotheses:
    - One that is instance of connective
    - Zero or more with proposition as conclusion, but some connective component as its hypotheses

Introduction Rules

Truth Introduction:

$$\frac{}{\mathbf{T}} \; \mathbf{T} \, \mathsf{I}$$

And Introduction:

$$\frac{A \quad B}{A \wedge B} \; \mathsf{And} \, \mathsf{I}$$

Or Introduction:

$$\frac{A}{A \vee B} \; \mathsf{Or}_L \, \mathsf{I}$$

$$\frac{B}{A \vee B} \; \mathsf{Or}_R \, \mathsf{I}$$

Not Introduction:

$$\begin{array}{c} A \\ \vdots \\ \dfrac{\mathbf{F}}{\neg A} \; \mathsf{Not} \, \mathsf{I} \end{array}$$

Implication Introduction:

$$\begin{array}{c} A \\ \vdots \\ \dfrac{B}{A \Rightarrow B} \; \mathsf{Imp} \, \mathsf{I} \end{array}$$

No False Introduction

$$\overline{A \Rightarrow (B \Rightarrow (A \wedge B))}$$

Example Proof 1

$$\frac{A}{}$$

$$\frac{\dfrac{\overline{}}{B \Rightarrow (A \wedge B)}}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

Example Proof 1

$$\cfrac{\cfrac{\cfrac{A \quad B}{A \wedge B}}{B \Rightarrow (A \wedge B)} \; \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))} \; \text{Imp I}$$

$$\cfrac{\cfrac{\dfrac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

Example Proof 1

$$\cfrac{\cfrac{\dfrac{A \quad B}{A \land B} \; \text{And I}}{B \Rightarrow (A \land B)} \; \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \land B))} \; \text{Imp I}$$

- All assumptions discharged; proof complete

$$\frac{}{B \Rightarrow (A \land B)}$$

Example Proof 2

$$\frac{\dfrac{B}{A \wedge B}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

Example Proof 2

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\ \text{And I}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}$$

Example Proof 2

$$\frac{\dfrac{A? \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

Example Proof 2

$$\frac{\dfrac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved "Assuming A, we have $B \Rightarrow (A \wedge B)$

Discharging Hypothesis

$$\overline{A \Rightarrow (A \wedge A)}$$

$$\dfrac{\dfrac{A \quad A}{A \land A}\text{ And I}}{A \Rightarrow (A \land A)}\text{ Imp I}$$

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

$$\dfrac{\dfrac{\dfrac{A \quad A}{A \wedge A}\ \text{And I}}{A \Rightarrow (A \wedge A)}\ \text{Imp I}}{} \qquad \dfrac{}{A \Rightarrow (B \Rightarrow A)}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I} \qquad\qquad \frac{\dfrac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I} \qquad \dfrac{\dfrac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I} \qquad\qquad \frac{\dfrac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

$$A \Rightarrow (A \lor B)$$

Elimination Rules

- So far, have rules to "introduce" logical connectives into propositions
- No rules for how to "use" logical connectives
  - No assumptions with logical connectives
- Need "elimination" rules
- Example: Can't prove

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

  with what we have so far
- Elimination rules assume assumption with a connective; have general conclusion
  - Generally needs additional hypotheses

Elimination Rules

False Elimination:

$$\frac{\mathbf{F}}{C} \; \mathbf{F} \; E$$

Not Elimination:

$$\frac{\neg A \quad A}{C} \; \text{Not E}$$

And Elimination:

$$\frac{A \wedge B \quad \begin{array}{c} A \\ \vdots \\ C \end{array}}{C} \; \text{And}_L \; E$$

$$\frac{A \wedge B \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \; \text{And}_R \; E$$

Or Elimination:

$$\frac{A \vee B \quad \begin{array}{c} A \\ \vdots \\ C \end{array} \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \; \text{Or E}$$

Implication Elimination:

$$\frac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \; \text{Imp E}$$

$$\overline{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

Example Proof 4

$$\frac{\overline{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \; \text{Imp I}$$

Example Proof 4

$$\cfrac{\cfrac{\overline{A \Rightarrow C}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}$$

$$\frac{\dfrac{\dfrac{\dfrac{C}{A \Rightarrow C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}}{}$$

Example Proof 4

$$\cfrac{A \Rightarrow B \quad A \quad \cfrac{\qquad\qquad}{C}}{\cfrac{\cfrac{\cfrac{C}{A \Rightarrow C}\ \text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\ \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}}\ \text{Imp E}$$

Example Proof 4

$$\frac{A \Rightarrow B \quad A \quad \dfrac{}{C}}{\dfrac{\dfrac{C}{A \Rightarrow C} \text{ Imp I}}{\dfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}} \text{ Imp E}$$

$$\cfrac{A \Rightarrow B \quad A \quad \cfrac{}{C}}{\cfrac{\cfrac{C}{A \Rightarrow C} \text{ Imp I}}{\cfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}} \text{ Imp E}$$

Example Proof 4

$$\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C} \; \text{Imp E}}{\dfrac{\dfrac{C}{A \Rightarrow C} \; \text{Imp I}}{\dfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \; \text{Imp I}} \; \text{Imp I}} \; \text{Imp E}$$

Example Proof 4

$$\cfrac{A \Rightarrow B \quad A \quad \cfrac{\cfrac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp E}}{\cfrac{\cfrac{\cfrac{C}{A \Rightarrow C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}}$$

Example Proof 4

$$\cfrac{A \Rightarrow B \quad A \qquad \cfrac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{\cfrac{\cfrac{C}{A \Rightarrow C} \text{ Imp I}}{\cfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}} \text{ Imp E}$$

Example Proof 4

$$\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{\dfrac{\dfrac{\dfrac{C}{A \Rightarrow C} \text{ Imp I}}{\dfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}}{} \text{ Imp E}}$$

Some Well-Known Derived Rules

Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B} \text{ MP}$$

$$\frac{A \Rightarrow B \quad A \quad B}{B} \text{ Imp E}$$

Left Conjunct

$$\frac{A \wedge B}{A} \text{ AndL}$$

$$\frac{A \wedge B \quad A}{A} \text{ And}_L \text{ E}$$

Right Conjunct

$$\frac{A \wedge B}{B} \text{ AndR}$$

$$\frac{A \wedge B \quad A}{A} \text{ And}_R \text{ E}$$

Your Turn

$$\frac{}{(A \wedge B) \Rightarrow (A \vee B)}$$

Assumptions in Natural Deduction

- Problem: Keeping track of hypotheses and their discharge in Natural Deduction is *HARD!*
- Solution: Use *sequents* to track hypotheses

- A sequent is a pair of
  - A set of propositions (called assumptions, or hypotheses of sequent) and
  - A proposition (called conclusion of sequent)
- More generally (not here), allow set of hypotheses and set of conclusions

Nat. Ded. Introduction Sequent Rules

Γ is set of propositions (assumptions/hypotheses)

Hypothesis Introduction:

$$\frac{}{\Gamma \cup \{A\} \vdash A} \; \text{Hyp}$$

Truth Introduction:

$$\frac{}{\Gamma \vdash \mathbf{T}} \; \mathbf{T} \; \text{I}$$

And Introduction:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \; \text{And I}$$

Or Introduction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \; \text{Or}_L \; \text{I}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \; \text{Or}_R \; \text{I}$$

Not Introduction:

$$\frac{\Gamma \cup \{A\} \vdash \mathbf{F}}{\Gamma \vdash \neg A} \; \text{Not I}$$

Implication Introduction:

$$\frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \Rightarrow B} \; \text{Imp I}$$

Nat. Ded. Elimination Sequent Rules

Γ is set of propositions (assumptions/hypotheses)

Not Elimination:

$$\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash C} \text{ Not E}$$

Implication Elimination:

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{ Imp E}$$

And Elimination:

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{A\} \vdash C}{\Gamma \vdash C} \text{ And}_L \text{ E}$$

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{ And}_R \text{ E}$$

False Elimination:

$$\frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash C} \mathbf{F} \text{ E}$$

Or Elimination:

$$\frac{\Gamma \vdash A \vee B \quad \Gamma \cup \{A\} \vdash C \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{ Or E}$$

$$\overline{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

$$\frac{\overline{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4, Revisited

$$\frac{\dfrac{}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}}{\dfrac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$

$$\cfrac{\cfrac{\cfrac{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C} \text{ Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$

$$\dfrac{\dfrac{\overline{\Gamma_3 \vdash A \Rightarrow B} \quad \overline{\Gamma_3 \vdash A} \quad \overline{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C} \text{ Imp E}}{\dfrac{\dfrac{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B, \ B \Rightarrow C, \ A\}$
$\Gamma_4 = \{A \Rightarrow B, \ B \Rightarrow C, \ A, \ B\}$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \cfrac{}{\Gamma_3 \vdash A} \quad \cfrac{}{\Gamma_4 \vdash C}
      }{\{A \Rightarrow B, \ B \Rightarrow C, \ A\} \vdash C} \text{Imp E}
    }{\{A \Rightarrow B, \ B \Rightarrow C\} \vdash A \Rightarrow C} \text{Imp I}
  }{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}
}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}
$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$

$$\cfrac{\cfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \cfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \cfrac{}{\Gamma_4 \vdash C}}{\cfrac{\cfrac{\cfrac{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C} \text{ Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp E}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$
\dfrac{
\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\dfrac{}{\Gamma_4 \vdash B \Rightarrow C} \quad \dfrac{}{\Gamma_4 \vdash B} \quad \dfrac{}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\ \text{Imp E}
}{
\dfrac{\dfrac{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}{\dfrac{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}{\dfrac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}}\ \text{Imp I}}\ \text{Imp I}
}\ \text{Imp E}
$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B, \ B \Rightarrow C, \ A\}$
$\Gamma_4 = \{A \Rightarrow B, \ B \Rightarrow C, \ A, \ B\}$
$\Gamma_5 = \{A \Rightarrow B, \ B \Rightarrow C, \ A, \ B, \ C\}$

$$
\cfrac{
  \cfrac{\cfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \qquad \cfrac{\text{Hyp}}{\Gamma_3 \vdash A} \qquad \cfrac{\cfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \qquad \cfrac{}{\Gamma_4 \vdash B} \qquad \cfrac{}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\ \text{Imp E}}{\{A \Rightarrow B, \ B \Rightarrow C, \ A\} \vdash C}\ \text{Imp E}
}{
  \cfrac{
    \cfrac{
      \{A \Rightarrow B, \ B \Rightarrow C\} \vdash A \Rightarrow C
    }{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\ \text{Imp I}
  }{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}
}\ \text{Imp I}
$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$
\cfrac{
  \cfrac{
    \cfrac{\cfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \cfrac{\text{Hyp}}{\Gamma_3 \vdash A}}{}
    \quad
    \cfrac{\cfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \quad \cfrac{\text{Hyp}}{\Gamma_4 \vdash B}}{\Gamma_4 \vdash C} \quad \cfrac{\text{Hyp}}{\Gamma_5 \vdash C} \ \text{Imp E}
  }{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C} \ \text{Imp E}
}{
  \cfrac{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}{\cfrac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \ \text{Imp I}} \ \text{Imp I}
} \ \text{Imp I}
$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$
\cfrac{
  \cfrac{\cfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B}\quad \cfrac{\text{Hyp}}{\Gamma_3 \vdash A}\quad \cfrac{\cfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C}\quad \cfrac{\text{Hyp}}{\Gamma_4 \vdash B}\quad \cfrac{\text{Hyp}}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\ \text{Imp E}}
  {
    \cfrac{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}
    {\cfrac{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}
    {\cfrac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}
    {\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}}\ \text{Imp I}}\ \text{Imp I}
  }
}{}\ \text{Imp E}
$$