CS477 Formal Software Development Methods

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Proofs in Propositional Logic

- Natural Deduction proofs are
  - trees with nodes that are inference rules
  - and a discharge function
- Inference rule has hypotheses and conclusion
- Conclusion: a single proposition
- Hypotheses: zero or more propositions, each possibly with hypotheses
  - Rule with no hypotheses called an axiom
- Inference rule graphically presents as

$$\dfrac{H_1 \quad \ldots \quad \begin{matrix} A_i \\ \vdots \\ H_i \end{matrix} \quad \ldots \quad H_j \quad \ldots \quad \begin{matrix} A_k \\ \vdots \\ H_k \end{matrix} \quad \ldots \quad H_n}{C}\ \textit{rule}$$

- Discharge function: maps proof tree leaves with node (induction rule) introducing it
- Instance of inference rule above may discharge A_i and A_j in a proof using *rule*

Inference Rules

- Inference rules associated with connectives
- Two main kinds of (non-axiom) inference rules:
  - Introduction – says how to conclude proposition made from connective is true
    - Proposition below the line is instance of the connective
    - Hypotheses are (some of) connective components
  - Eliminations – says how to use a proposition made from connective to prove result
  - Conclusion: propositional atom (*i.e.* any conclusion can be instance)
  - Hypotheses:
    - One that is instance of connective
    - Zero or more with proposition as conclusion, but some connective component as its hypotheses

Introduction Rules

Truth Introduction:

$$\dfrac{}{\mathbf{T}}\ \mathbf{T}\ \mathrm{I}$$

And Introduction:

$$\dfrac{A \quad B}{A \wedge B}\ \text{And I}$$

Or Introduction:

$$\dfrac{A}{A \vee B}\ \text{Or}_L\ \text{I} \qquad \dfrac{B}{A \vee B}\ \text{Or}_R\ \text{I}$$

Not Introduction:

$$\dfrac{\begin{matrix} A \\ \vdots \\ \mathbf{F} \end{matrix}}{\neg A}\ \text{Not I}$$

Implication Introduction:

$$\dfrac{\begin{matrix} A \\ \vdots \\ B \end{matrix}}{A \Rightarrow B}\ \text{Imp I}$$

No False Introduction

Example Proof 1

$$\dfrac{}{A \Rightarrow (B \Rightarrow (A \wedge B))}$$

Example Proof 1

$$\dfrac{\dfrac{\dfrac{A}{}}{B \Rightarrow (A \wedge B)}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

Example Proof 1

$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

Example Proof 1

$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}\ \text{And I}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

Example Proof 1

$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}\ \text{And I}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

- All assumptions discharged; proof complete

Example Proof 2

$$\frac{}{B \Rightarrow (A \wedge B)}$$

Example Proof 2

$$\frac{\dfrac{B}{A \wedge B}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}$$

Example Proof 2

$$\frac{\dfrac{A \quad B}{A \wedge B}\ \text{And I}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}$$

Example Proof 2

$$\dfrac{\dfrac{A? \quad B}{A \wedge B}\text{And I}}{B \Rightarrow (A \wedge B)}\text{Imp I}$$

Example Proof 2

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\text{And I}}{B \Rightarrow (A \wedge B)}\text{Imp I}$$

- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved "Assuming A, we have $B \Rightarrow (A \wedge B)$

Discharging Hypothesis

$$\dfrac{}{A \Rightarrow (A \wedge A)}$$

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\text{And I}}{A \Rightarrow (A \wedge A)}\text{Imp I}$$

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\text{And I}}{A \Rightarrow (A \wedge A)}\text{Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\text{And I}}{A \Rightarrow (A \wedge A)}\text{Imp I} \qquad \dfrac{}{A \Rightarrow (B \Rightarrow A)}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \dfrac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \dfrac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \dfrac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

Your Turn

$$\dfrac{\qquad\qquad\qquad}{A \Rightarrow (A \vee B)}$$

Elimination Rules

- So far, have rules to "introduce" logical connectives into propositions
- No rules for how to "use" logical connectives
  - No assumptions with logical connectives
- Need "elimination" rules
- Example: Can't prove

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

  with what we have so far
- Elimination rules assume assumption with a connective; have general conclusion
  - Generally needs additional hypotheses

Elimination Rules

False Elimination:

$$\dfrac{\mathbf{F}}{C}\,\mathbf{F}\text{ E}$$

Not Elimination:

$$\dfrac{\neg A \quad A}{C}\,\text{Not E}$$

And Elimination:

$$\dfrac{A \wedge B \quad \begin{array}{c} A \\ \vdots \\ C \end{array}}{C}\,\text{And}_L\text{ E} \qquad \dfrac{A \wedge B \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C}\,\text{And}_R\text{ E}$$

Or Elimination:

$$\dfrac{A \vee B \quad \begin{array}{c} A \\ \vdots \\ C \end{array} \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C}\,\text{Or E}$$

Implication Elimination:

$$\dfrac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C}\,\text{Imp E}$$

Example Proof 4

$$\frac{}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

Example Proof 4

$$\frac{\frac{}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{}{A \Rightarrow C}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{\frac{}{C}}{A \Rightarrow C}\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{}{C}}{C}\text{Imp E}}{A \Rightarrow C}\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{}{C}}{C}\text{Imp E}}{A \Rightarrow C}\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{}{C}}{C}\;\text{Imp E}}{A \Rightarrow C}\;\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\;\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\;\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C}\;\text{Imp E}}{C}\;\text{Imp E}}{A \Rightarrow C}\;\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\;\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\;\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C}\;\text{Imp E}}{C}\;\text{Imp E}}{A \Rightarrow C}\;\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\;\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\;\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C}\;\text{Imp E}}{C}\;\text{Imp E}}{A \Rightarrow C}\;\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\;\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\;\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C}\;\text{Imp E}}{C}\;\text{Imp E}}{A \Rightarrow C}\;\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\;\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\;\text{Imp I}$$

Some Well-Known Derived Rules

Modus Ponens

$$\dfrac{A \Rightarrow B \quad A}{B}\;\text{MP} \qquad\qquad \dfrac{A \Rightarrow B \quad A \quad B}{B}\;\text{Imp E}$$

Left Conjunct

$$\dfrac{A \wedge B}{A}\;\text{AndL} \qquad\qquad \dfrac{A \wedge B \quad A}{A}\;\text{And}_L\text{ E}$$

Right Conjunct

$$\dfrac{A \wedge B}{B}\;\text{AndR} \qquad\qquad \dfrac{A \wedge B \quad A}{A}\;\text{And}_R\text{ E}$$

Your Turn

$$\frac{}{(A \wedge B) \Rightarrow (A \vee B)}$$

Assumptions in Natural Deduction

- Problem: Keeping track of hypotheses and their discharge in Natural Deduction is *HARD!*
- Solution: Use *sequents* to track hypotheses
- A sequent is a pair of
  - A set of propositions (called assumptions, or hypotheses of sequent) and
  - A proposition (called conclusion of sequent)
- More generally (not here), allow set of hypotheses and set of conclusions

Nat. Ded. Introduction Sequent Rules

Γ is set of propositions (assumptions/hypotheses)
Hypothesis Introduction:

$$\frac{}{\Gamma \cup \{A\} \vdash A}\ \text{Hyp}$$

Truth Introduction:

$$\frac{}{\Gamma \vdash \mathbf{T}}\ \mathbf{T}\ \text{I}$$

And Introduction:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \text{And I}$$

Or Introduction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \text{Or}_L\ \text{I} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \text{Or}_R\ \text{I}$$

Not Introduction:

$$\frac{\Gamma \cup \{A\} \vdash \mathbf{F}}{\Gamma \vdash \neg A}\ \text{Not I}$$

Implication Introduction:

$$\frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \Rightarrow B}\ \text{Imp I}$$

Nat. Ded. Elimination Sequent Rules

Γ is set of propositions (assumptions/hypotheses)
Not Elimination:

$$\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash C}\ \text{Not E}$$

Implication Elimination:

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C}\ \text{Imp E}$$

And Elimination:

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{A\} \vdash C}{\Gamma \vdash C}\ \text{And}_L\ \text{E} \qquad \frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C}\ \text{And}_R\ \text{E}$$

False Elimination:

$$\frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash C}\ \mathbf{F}\ \text{E}$$

Or Elimination:

$$\frac{\Gamma \vdash A \vee B \quad \Gamma \cup \{A\} \vdash C \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C}\ \text{Or E}$$

Example Proof 4, Revisited

$$\frac{}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

Example Proof 4, Revisited

$$\frac{\dfrac{}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}$$

Example Proof 4, Revisited

$$\dfrac{\dfrac{\dfrac{}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{}{\Gamma_3 \vdash A} \quad \dfrac{}{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{}{\Gamma_3 \vdash A} \quad \dfrac{}{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{}{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\dfrac{}{\Gamma_4 \vdash B \Rightarrow C} \quad \dfrac{}{\Gamma_4 \vdash B} \quad \dfrac{}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \quad \dfrac{}{\Gamma_4 \vdash B} \quad \dfrac{}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \quad \dfrac{\text{Hyp}}{\Gamma_4 \vdash B} \quad \dfrac{}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \quad \dfrac{\text{Hyp}}{\Gamma_4 \vdash B} \quad \dfrac{\text{Hyp}}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\,\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\,\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{\{\ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$