CS477 Formal Software Development Methods

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Semantics of Propositional Logic: Model Theory

Model for Propositional Logic has three parts

- Mathematical set of values used as meaning of propositions
- Interpretation function giving meaning to props built from logical connectives, via structural recursion

Standard Model of Propositional Logic

- $\mathcal{B} = \{\mathrm{true}, \mathrm{false}\}$ boolean values
- $v : AP \to \mathcal{B}$ a valuation
- Interpretation function ...

Semantics of Propositional Logic: Model Theory

Standard Model of Propositional Logic (cont)

- Standard interpretation $\mathcal{I}_v$ defined by structural induction on formulae:
  - $\mathcal{I}_v(\mathbf{T}) = \mathrm{true}$ and $\mathcal{I}_v(\mathbf{F}) = \mathrm{false}$
  - If $a \in AP$ then $\mathcal{I}_v(a) = v(a)$
  - For $p \in PROP$, if $\mathcal{I}_v(p) = \mathrm{true}$ then $\mathcal{I}_v(\neg p) = \mathrm{false}$, and if $\mathcal{I}_v(p) = \mathrm{false}$ then $\mathcal{I}_v(\neg p) = \mathrm{true}$
  - For $p,\ q \in PROP$
    - If $\mathcal{I}_v(p) = \mathrm{true}$ and $\mathcal{I}_v(q) = \mathrm{true}$, then $\mathcal{I}_v(p \wedge q) = \mathrm{true}$, else $\mathcal{I}_v(p \wedge q) = \mathrm{false}$
    - If $\mathcal{I}_v(p) = \mathrm{true}$ or $\mathcal{I}_v(q) = \mathrm{true}$, then $\mathcal{I}_v(p \vee q) = \mathrm{true}$, else $\mathcal{I}_v(p \vee q) = \mathrm{false}$
    - If $\mathcal{I}_v(q) = \mathrm{true}$ or $\mathcal{I}_v(p) = \mathrm{false}$, then $\mathcal{I}_v(p \Rightarrow q) = \mathrm{true}$, else $\mathcal{I}_v(p \Rightarrow q) = \mathrm{false}$
    - If $\mathcal{I}_v(p) = \mathcal{I}_v(q)$ then $\mathcal{I}_v(p \Leftrightarrow q) = \mathrm{true}$, else $\mathcal{I}_v(p \Leftrightarrow q) = \mathrm{false}$

Truth Tables

Interpretation function often described by truth table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
true	true					
true	false					
false	true					
false	false					

Truth Tables

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true	true	false				
true	false	false				
false	true	true				
false	false	true				

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false	true	true	false	true	true	
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true	false	false	false	true	false	false
false	true	true	false	true	true	false
false	false	true	false	false	true	true

Modeling Propositional Formulae

- $(\mathcal{B}, \mathcal{I})$ is the standard model of proposition logic
- Given valuation v and proposition $p \in PROP$, write $v \models p$ iff $\mathcal{I}_v(p) = \mathrm{true}$
  - More fully written as $\mathcal{B}, \mathcal{I}, v \models p$
  - Say v satisfies p, or v models p
  - Write $v \not\models p$ if $\mathcal{I}_v(p) = \mathrm{false}$
- p is satisfiable if there exists valuation v such that $v \models p$
- p is valid, a.k.a. a tautology if for every valuation v we have $v \models p$
- p is logically equivalent to q, $p \equiv q$ if for every valuation, v, we have $v \models p$ iff $v \models q$
  - Claim: Logical equivalence is an equivalence relation

Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true			
true	false			
false	true			
false	false			

Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true	true		
true	false	false		
false	true	true		
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Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true	true	true	
true	false	false	true	
false	true	true	true	
false	false	true	false	

Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	true	true	true	true
true	false	false	true	true
false	true	true	true	true
false	false	true	false	true

Example Tautology – Your Turn

Example: Logical Equivalence

$$A \Rightarrow B \equiv ((\neg A) \vee B)$$

A	B	$A \Rightarrow B$	$\neg A$	$(\neg A) \vee B$
true	true	true	false	true
true	false	false	false	false
false	true	true	true	true
false	false	true	true	true

More Useful Logical Equivalences

$$\neg\neg A \equiv A \qquad \neg \mathbf{T} \equiv \mathbf{F} \qquad \neg \mathbf{F} \equiv \mathbf{T}$$
$$(A \vee A) \equiv A \qquad (A \vee B) \vee C \equiv A \vee (B \vee C)$$
$$(A \wedge A) \equiv A \qquad (A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$
$$A \vee B \equiv B \vee A \qquad \neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$$
$$A \wedge B \equiv B \wedge A \qquad \neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$$
$$(A \wedge \neg A) \equiv \mathbf{F} \qquad A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$
$$(A \vee \neg A) \equiv \mathbf{T} \qquad (A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$$
$$(\mathbf{T} \wedge A) \equiv A \qquad A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$
$$(\mathbf{T} \vee A) \equiv \mathbf{T} \qquad (A \wedge B) \vee C \equiv (A \wedge C) \vee (B \wedge C)$$
$$(\mathbf{F} \wedge A) \equiv \mathbf{F} \qquad (\mathbf{F} \vee A) \equiv A$$

Logical Equivalence a Structural Congruence

Theorem

Logical equivalence is a structural congruence. That is, if $p \equiv p'$ and $q \equiv q'$ then

1. $\neg p \equiv \neg p'$
2. $p \wedge q \equiv p' \wedge q'$
3. $p \vee q \equiv p' \vee q'$
4. $p \Rightarrow q \equiv p' \Rightarrow q'$
5. $p \Leftrightarrow q \equiv p' \Leftrightarrow q'$

Logical Equivalence a Structural Congruence

Proof.

- Assume $p \equiv p'$ and $q \equiv q'$
- **Hyp**: Then for all valuations v, $v \models p$ iff $v \models p'$ and $v \models q$ iff $v \models q'$, i.e. $\mathcal{I}_v(p) = \text{true}$ iff $\mathcal{I}_v(p') = \text{true}$ and $\mathcal{I}_v(q) = \text{true}$ iff $\mathcal{I}_v(q') = \text{true}$
- Case 4: Show $p \Rightarrow q \equiv p' \Rightarrow q'$
  - Other cases done same way
- Need to show for all v, $\mathcal{I}_v(p \Rightarrow q) = \text{true}$ iff $\mathcal{I}_v(p' \Rightarrow q') = \text{true}$
- Fix v
- Need to show if $\mathcal{I}_v(p \Rightarrow q) = \text{true}$ then $\mathcal{I}_v(p' \Rightarrow q') = \text{true}$, and if $\mathcal{I}_v(p' \Rightarrow q') = \text{true}$ then $\mathcal{I}_v(p \Rightarrow q) = \text{true}$

□

Logical Equivalence a Structural Congruence

Proof.

- $(\Longrightarrow)$
  - Assume $\mathcal{I}_v(p \Rightarrow q) = \text{true}$
  - By closure property of inductive definition of $\mathcal{I}$, either $\mathcal{I}_v(q) = \text{true}$ or $\mathcal{I}_v(p) = \text{false}$.
  - Therefore, by **Hyp**, either $\mathcal{I}_v(q') = \text{true}$ or $\mathcal{I}_v(p') = \text{false}$
    - since $\mathcal{B}$ has only two elements, and $\mathcal{I}_v$ total (proof?)
  - By $\mathcal{I}$ def, have $\mathcal{I}_v(p' \Rightarrow q')$
- $(\Longleftarrow)$

□

Non-standard Model of Propositional Logic

Other models possible
Example:

- $\mathcal{C} = \{\text{true}, \text{false}, \bot\}$
- Valuations assign values in cC to propositional atoms
- If $\mathcal{J}_w(p) = \bot$ then $\mathcal{J}_w(\neg p) = \bot$, otherwise same as for $\mathcal{I}$
- $\mathcal{J}_w(p) = bot$ or $\mathcal{J}_w(q) = \bot$ then $\mathcal{J}_w(\neg p) = \bot$, $\mathcal{J}_w(p \wedge q) = \bot$, $\mathcal{J}_w(p \vee q) = \bot$, $\mathcal{J}_w(p \Rightarrow q) = \bot$, and $\mathcal{J}_w(p \Leftrightarrow q) = \bot$; otherwise same as for $\mathcal{I}$
- Note: $A \vee \neg A \not\equiv \mathbf{T}$

Proofs in Propositional Logic

- Natural Deduction proofs are trees with nodes that are inference rules
- Inference rule has hypotheses and conclusion
- Conclusion a single proposition
- Hypotheses zero or more propositions, possibly with hypotheses
- Two main kinds of inference rules:
  - Introduction – says how to conclude proposition made from connective is true
  - Eliminations – says how to use a proposition made from connective to prove result
- Inference rules associated with connectives
- Rule with no hypotheses called an axiom

Introduction Rules

Truth Introduction:

$$\frac{}{\mathbf{T}}\ \mathbf{T}\ \text{I}$$

And Introduction:

$$\frac{A \quad B}{A \wedge B}\ \text{And I} A \wedge B$$

Or Introduction:

$$\frac{A}{A \vee A \vee B}\ \text{Or}_L\ \text{I} \qquad \frac{A \vee B}{B \vee A \vee B}\ \text{Or}_R\ \text{I}$$

Not Introduction:

$$\frac{\begin{matrix} A \\ \vdots \\ \mathbf{F} \end{matrix}}{\neg A}\ \text{Not I}$$

Implication Introduction:

$$\frac{\begin{matrix} A \\ \vdots \\ B \end{matrix}}{A \Rightarrow B}\ \text{Imp I}$$

No False Introduction

Example Proof 1

$$\frac{}{A \Rightarrow (B \Rightarrow (A \wedge B))}$$

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$$\cfrac{\cfrac{\cfrac{A}{\quad}}{B \Rightarrow (A \wedge B)}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

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$$\cfrac{\cfrac{\cfrac{A \quad B}{A \wedge B}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

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Example Proof 1

$$\cfrac{\cfrac{\cfrac{A \quad B}{A \wedge B}\ \text{And I}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \text{Imp I}$$

- All assumptions discharged; proof complete

Example Proof 2

$$\cfrac{\quad}{B \Rightarrow (A \wedge B)}$$

Example Proof 2

$$\cfrac{\cfrac{B}{A \wedge B}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}$$

Example Proof 2

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\text{And I}}{B \Rightarrow (A \wedge B)}\text{Imp I}$$

Example Proof 2

$$\dfrac{\dfrac{A? \quad B}{A \wedge B}\text{And I}}{B \Rightarrow (A \wedge B)}\text{Imp I}$$

Example Proof 2

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\text{And I}}{B \Rightarrow (A \wedge B)}\text{Imp I}$$

- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved "Assuming A, we have $B \Rightarrow (A \wedge B)$

Discharging Hypothesis

$$\dfrac{}{A \Rightarrow (A \wedge A)}$$

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- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \frac{}{A \Rightarrow (B \Rightarrow A)}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \frac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

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Discharging Hypothesis

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Discharging Hypothesis

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- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

Your Turn

$$\frac{}{A \Rightarrow (A \vee B)}$$