Program Verification: Lecture 25

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Extending Narrowing-Based Symbolic Model Checking

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Before answering these two questions (in the positive), this lecture first introduces some symbolic techniques needed for this purpose.

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Given a rewrite theory $\mathcal{R} = (\Sigma, E \cup B, R)$, and a term $t \in T_{\Sigma}(X)$, an R-narrowing step modulo $E \cup B$, denoted $t \rightsquigarrow^{\theta}_{R,E \cup B} v$ holds iff there exists a non-variable position p in t, a rule $l \rightarrow r$ in R, and a $E \cup B$-unifier $\theta \in Unif_{E \cup B}(t|_p = l)$ such that $v = t[r]_p\theta$.

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But the million-dolar question is: How do we compute a complete set $\mathit{Unif}_{E \cup B}(t|_p = l)$ of $E \cup B$-unifiers?

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For $E \cup B$ an arbitrary set of equations $E \cup B$, computing such a set $Unif_{E \cup B}(u = v)$ is a very complex matter. But for our purposes we may assume that the oriented equations $\vec{E}$ are convergent modulo B, which makes the task much easier.

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1. $\Sigma^{\equiv}$ extends Σ by adding: (a) for each connected component $[s]$ in Σ not having a top sort $\top_{[s]}$, such a new top sort $\top_{[s]}$; (b) a new sort *Pred* with a constant *tt*; and (c) for each connected component $[s]$ in Σ a binary equality predicate $\_ \equiv \_ : \top_{[s]}\ \top_{[s]} \to Pred$.

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2. $E^{\equiv}$ extends E by adding for each connected component $[s]$ in Σ an equation $x{:}\top_{[s]} \equiv x{:}\top_{[s]} = tt$.

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It is easy to check (exercise!) that if $\vec{E}$ is convergent modulo B, then $\vec{E}^{\equiv}$ is convergent modulo B. But then (†) becomes:

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Theorem. θ is a $E \cup B$-unifier of $u = v$ iff $(u\theta \equiv v\theta)!_{\vec{E}^{\equiv}/B} = tt$.

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This gives us our desired $E \cup B$-unification semi-algorithm, whose proof of correctness follows easily (exercise!) by repeated application of the Lifting Lemma for the rewrite theory $(\Sigma^{\equiv}, B, \vec{E}^{\equiv})$, just by observing that θ is a $E \cup B$-unifier of $u = v$ iff its $\vec{E}/B$-normalized form $\theta!_{\vec{E}/B}$ is so.

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For narrowing-based model checking, we obtain as an immediate corollary the following vast generalization of the Completeness of Narrowing Search Theorem in Lecture 21 for topmost theories:

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The proof, by applying the Lifting Lemma, generalizes the similar proof in Lecture 21 and is left as an exercise.

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In the above, generalized Completeness of Narrowing Search Theorem, narrowing happens at two levels: (i) with R modulo $E \cup B$ for reachability analysis, and (ii) with $\vec{E}^{\equiv}$ modulo B for computing $E \cup B$-unifiers.

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To overcome these performance barriers, the technique of folding an infinite narrowing tree into a (hopefully finite) narrowing graph can be applied at both levels. For the symbolic reachability level with $\leadsto^{*}_{R,(E \cup B)}$ we have already seen this in Lecture 21. Likewise, for $\vec{E}$, B-narrowing with $\vec{E}$ convergent modulo B ($\vec{E}^{\equiv}$, B-narrowing is just a special case), folding variant narrowing delivers the goods:

Folding Variant Narrowing

Folding Variant Narrowing, proposed by S. Escobar, R. Sasse and J. Meseguer[1] for theories $(\Sigma, E \cup B)$ with $\vec{E}$ convergent modulo B, folds the $\vec{E}, B$-narrowing tree of t into a graph in a breadth first manner as follows:

[1] "Folding variant narrowing and optimal variant termination", J. Alg. & Log. Prog., 81, 898–928, 2012.

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1. It considers only paths $t \overset{\theta}{\rightsquigarrow}{}^{n}_{\vec{E},B}\ u$ in the narrowing tree such that u and θ are $\vec{E}, B$-normalized.

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1. It considers only paths $t \overset{\theta}{\leadsto}{}^{n}_{\vec{E},B}\ u$ in the narrowing tree such that u and θ are $\vec{E}, B$-normalized.
2. For any such path $t \overset{\theta}{\leadsto}{}^{n}_{\vec{E},B}\ u$, if there is another such different path $t \overset{\theta'}{\leadsto}{}^{m}_{\vec{E},B}\ u'$ with $m \leq n$ and a B-matching substitution γ such that: (i) $u =_B u'\gamma$, and (ii) $\theta =_B \theta'\gamma$, then the node u is folded into the more general node u'.

[1] "Folding variant narrowing and optimal variant termination", J. Alg. & Log. Prog., 81, 898–928, 2012.

Folding Variant Narrowing (II)

The pairs (u, θ) associated to paths $t \overset{\theta}{\rightsquigarrow}^{n}_{\vec{E},B} u$ in such a graph are called the $\vec{E}, B$-variants of t; and the graph thus obtained is called the folding variant narrowing graph of t.

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Maude supports the enumeration of all variants in the folding variant narrowing graph of t by the `get variants` t `.` command (§14.4, Maude Manual). It also supports variant-based $E \cup B$-unification when $\vec{E}$ is convergent modulo B with the `variant unify` command (§14.9, Maude Manual).

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$(\Sigma, E \cup B)$ enjoys the finite variant property (FVP) iff for any Σ-term t its folding variant graph is finite. This property holds iff for each $f : s_1 \ldots s_n \rightarrow s$ in Σ the folding variant graph of $f(x_1 : s_1, \ldots, x_n : s_n)$ is finite, which can be checked in Maude.

An FVP Example: SET

In the theory $(\Sigma, E \cup AC)$ SET below we can preform AC-unification in Maude as follows:

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```
fmod SET is
sort Set .
ops mt a b c d e f g : -> Set [ctor] .
op _U_ : Set Set -> Set [ctor assoc comm] . *** union
vars S S' : Set .
eq S U mt = S [variant] .           *** identity
eq S U S = S [variant] .            *** idempotencu
eq S U S U S' = S U S' [variant] . *** idempotency extension
endfm

unify a U a U b U S =? a U c U S' .

Unifier 1
S --> c U #1:Set
S' --> a U b U #1:Set

Unifier 2
S --> c
S' --> a U b
```

An FVP Example: SET (II)

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SET is FVP because `S U S'` has a finite number of variants:

```
get variants S U S' .

Variant 1
Set: #1:Set U #2:Set
S --> #1:Set
S' --> #2:Set

Variant 2
Set: %1:Set
S --> mt
S' --> %1:Set

Variant 3
Set: %1:Set
S --> %1:Set
S' --> mt

Variant 4
Set: %1:Set
S --> %1:Set
S' --> %1:Set
```

An FVP Example: SET (III)

```
Variant 5
Set: %1:Set U %2:Set U %3:Set
S --> %1:Set U %2:Set
S' --> %1:Set U %3:Set

Variant 6
Set: %1:Set U %2:Set
S --> %1:Set U %2:Set
S' --> %2:Set

Variant 7
Set: %1:Set U %2:Set
S --> %2:Set
S' --> %1:Set U %2:Set

No more variants.
```

Variant Unification for FVP Theories

It is easy to check (exercise!) that if $(\Sigma, E \cup B)$ is FVP, then $(\Sigma^{\equiv}, E^{\equiv} \cup B)$ is also FVP. This means that, when $(\Sigma, E \cup B)$ is FVP, variant unification always provides a finite and complete set of $E \cup B$-unifiers. For example, since SET is FVP any $E \cup AC$-unification problem has a finite number of variant unifiers.

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```
filtered variant unify a U a U b U S =? a U c U S' .

Unifier 1
S --> c U %1:Set
S' --> b U %1:Set

Unifier 2
S --> a U c U #1:Set
S' --> b U #1:Set

Unifier 3
S --> c U #1:Set
S' --> a U b U #1:Set

No more unifiers.
```

Symbolic Model Checking for $\mathcal{R} = (\Sigma, E \cup B, R)$ when $E \cup B$ is FVP

Thus, for $(\Sigma, E \cup B)$ FVP, the Completeness of Narrowing Search Theorem for a rewrite theory $\mathcal{R} = (\Sigma, E \cup B, R)$ of pg. 8 makes symbolic model checking tractable. In fact, it is supported by the same `fvu-narrow` command already discussed in Lecture 21.

Symbolic Model Checking for $\mathcal{R} = (\Sigma, E \cup B, R)$ when $E \cup B$ is FVP

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In summary, we have generalized the symbolic model checking results from Lecture 21 to:

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Thus, for $(\Sigma, E \cup B)$ FVP, the Completeness of Narrowing Search Theorem for a rewrite theory $\mathcal{R} = (\Sigma, E \cup B, R)$ of pg. 8 makes symbolic model checking tractable. In fact, it is supported by the same `fvu-narrow` command already discussed in Lecture 21.

In summary, we have generalized the symbolic model checking results from Lecture 21 to: (i) any topmost rewrite theory $\mathcal{R} = (\Sigma, E \cup B, R)$ with $\vec{E}$ convergent modulo B, and (ii) made it tractable when $E \cup B$ is FVP. For symbolic model checking examples when $E \cup B$ is FVP, see §15 of the The Maude Manual. Further examples will be given in Lectures 26 and 27.

The Folding Narrowing Forest $FNF_{\mathcal{R}}(u_1 \vee \ldots \vee u_n)$

For $\mathcal{R} = (\Sigma, E \cup B, R)$ with $E \cup B$ FVP, the folding narrowing forest from $u_1 \vee \ldots \vee u_n$ is the forest $FNF_{\mathcal{R}}(u_1 \vee \ldots \vee u_n) =_{def} \bigcup_{n \in \mathbb{N}} FNF^n_{\mathcal{R}}(u_1 \vee \ldots \vee u_n)$,

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- $FNF^{n+1}_{\mathcal{R}}(u_1 \vee \ldots \vee u_n)$ has

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$$\{v \in prefront(FNF^{n+1}_{\mathcal{R}}(u_1 \vee \ldots \vee u_n)) \mid \nexists w \in FNF^n_{\mathcal{R}}(u_1 \vee \ldots \vee u_n)\ s.t.\ v \sqsubseteq_{E \cup B} w\}$$

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where $v \sqsubseteq_{E \cup B} w \quad \Leftrightarrow_{def} \quad \exists \theta \ \ s.t. \ \ v =_{E \cup B} w\theta$, is called the folding or subsumption or matching relation modulo $E \cup B$.

The Folding Narrowing Forest $FNF_{\mathcal{R}}(u_1 \vee \ldots \vee u_n)$ (II)

As an optimization, whenever $v, v' \in \mathit{front}(FNF_{\mathcal{R}}^{n+1}(u_1 \vee \ldots \vee u_n))$ are such that $v \sqsubseteq_{E \cup B} v'$ we can remove node v as redundant.

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We add to $FNF_{\mathcal{R}}^{n+1}(u_1 \vee \ldots \vee u_n)$ as new edges those narrowings $u \leadsto_{R,(E \cup B)} v$ s.t. $u \in front(FNF_{\mathcal{R}}^{n}(u_1 \vee \ldots \vee u_n))$ and $v \in front(FNF_{\mathcal{R}}^{n+1}(u_1 \vee \ldots \vee u_n))$.

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If for some $n \in \mathbb{N}$ $\mathit{front}(FNF_{\mathcal{R}}^{n}(u_1 \vee \ldots \vee u_n)) = \emptyset$, then we have $FNF_{\mathcal{R}}(u_1 \vee \ldots \vee u_n) = FNF_{\mathcal{R}}^{n}(u_1 \vee \ldots \vee u_n)$,

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The proof is an easy induction on k for narrowing sequences $u_i \rightsquigarrow^{k}_{R,(E \cup B)} v$, $1 \leq i \leq n$, using that $v \sqsubseteq_{E \cup B} w \Rightarrow [\![v]\!] \subseteq [\![w]\!]$.

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Theorem (Completeness of Folding Narrowing Search). For a topmost and admissible $\mathcal{R} = (\Sigma, E \cup B, R)$ with $E \cup B$ FVP, and $u_1 \vee \ldots \vee u_n$ and $v_1 \vee \ldots \vee v_m$ non-variable constructor patterns,

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