Appendix 1 to Lecture 21: Proof of the Lifting Lemma

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The Lifting Lemma states:

Theorem (Lifting Lemma). Let (Σ, R) be a term rewriting system, $t \in T_\Sigma(X)$, and θ an R-*irreducible* substitution (i.e., if $x \in dom(\theta)$, then $\theta(x)$ cannot be rewritten with R). Then for each rewrite step $t\theta \to_R u$ there is a narrowing step $t \overset{\alpha}{\leadsto}_R v$ and an R-irreducible substitution δ such that $v\delta = u$.

Proof: Since we have a rewrite step $t\theta \to_R u$ and θ is R-irreducible, the rewrite must happen at a non-variable position p of t. Therefore, there is a rule $l \to r$ in R and a substitution γ of the variables of l such that $t|_p\theta = l\gamma$ and $u = t\theta[r\gamma]_p$. Since without loss of generality[1] we may assume that t and $t\theta$ do not share any variables with l, we can rephrase the equality $t|_p\theta = l\gamma$ as, $t|_p(\theta \uplus \gamma) = l(\theta \uplus \gamma)$, which shows that $(\theta \uplus \gamma)$ is a unifier of the equation $t|_p = l$. For the same reason we have $u = t\theta[r\gamma]_p = t[r]_p(\theta \uplus \gamma)$. Therefore, there is a unifier α in the set $Unif(t|_p = l)$ and a substitution δ such that $(\theta \uplus \gamma) = \alpha\delta$. But this means that we have a narrowing step with rule $l \to r$ at positon p in t of the form, $t \overset{\alpha}{\leadsto}_R v$ with $v = t[r\gamma]_p\alpha$. Therefore, from $(\theta \uplus \gamma) = \alpha\delta$, we immediately get $v\delta = u$, as desired. The only pending issue is to check that δ is R-irreducible. But since we have $t|_p\alpha = l\alpha$ and, without loss of generality, we may assume that the domain of δ is[2] $rng(\alpha)$, assuming that δ is R-reducible exactly means that there is a variable x in $t|_p$ and a variable y in $t|_p\alpha$ such that $\delta(y)$ is R-reducible. But this is impossible, because $t|_p\theta = t|_p\alpha\delta$ and θ is R-irreducible by hypothesis. □

[1] Just by renaming the variables of l (and therefore those of r) with fresh new variables.

[2] For the definition of $rng(\alpha)$ see page 5 of Lecture 21's slides.