Appendix to Lecture 13: Proof of the Completeness Theorem

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Equation Sets. For $\Sigma = ((S, \leq), \Sigma)$ and order-sorted signature, define the set of Σ-equations in the obvious way (where X has a countably infinite set X_s of variables for each sort $s \in S$):

$$\Sigma.Eq = \{u = v \mid \exists s, s' \in S.\ u \in T_\Sigma(X)_s \wedge v \in T_\Sigma(X)_{s'} \wedge [s] = [s']\}.$$

Provable Theorems and Theorems. Given any set of Σ-equations $E \subseteq \Sigma.Eq$, define the set of its *provable theorems* as:

$$PThm(E) = \{u = v \in \Sigma.Eq \mid u =_E v\}.$$

Likewise, for any $E \subseteq \Sigma.Eq$, define the set of its (semantically true) *theorems* as:

$$Thm(E) = \{u = v \in \Sigma.Eq \mid \forall \mathbb{A} \in \mathbf{Alg}_{(\Sigma,E)},\ \mathbb{A} \models u = v\}.$$

The **Soundness Theorem** for equational logic states the inclusion $PThm(E) \subseteq Thm(E)$, and the **Completeness Theorem** states the opposite inclusion $Thm(E) \subseteq PThm(E)$. The goal of this Addendum is to prove the **Completeness Theorem**.

For any Σ-algebra $\mathbb{A}$ define its set of *semantic theorems* (i.e., equations that are *true* in $\mathbb{A}$) as:

$$Thm(\mathbb{A}) = \{u = v \in \Sigma.Eq \mid \mathbb{A} \models u = v\}.$$

Note that for each Σ-algebra $\mathbb{A}$ such that $\mathbb{A} \in \mathbf{Alg}_{(\Sigma,E)}$ we have the inclusions:

$$(\dagger) \quad PThm(E) \subseteq Thm(E) \subseteq Thm(\mathbb{A})$$

since the first inclusion is the already proved **Soundness Theorem**, and the second follows for the definitions of $Thm(E)$ and $Thm(\mathbb{A})$, plus the fact that $\mathbb{A} \in \mathbf{Alg}_{(\Sigma,E)}$.

Theorem (Completeness of Equational Logic). For Σ sensible, kind complete, and with non-empty sorts, and (Σ, E) an equational theory, we have the inclusion[1] $Thm(E) \subseteq PThm(E)$.

Proof. The two inclusions in $(\dagger)$ will collapse into equalities, thus proving the Theorem, if we can find a (Σ, E)-algebra $\mathbb{A}$ such that $Thm(\mathbb{A}) \subseteq PThm(E)$. But such a (Σ, E)-algebra $\mathbb{A}$ can be chosen to be $\mathbb{T}_{\Sigma/E}(X)$, where, by definition, $\mathbb{T}_{\Sigma/E}(X) = \mathbb{T}_{\Sigma(X)/E}|_\Sigma$, and X has a countably infinite set X_s of variables for each sort $s \in S$. Since we have proved that initial (Σ, E)-algebras satisfy the equations E, in particular this holds for initial $(\Sigma(X), E)$-algebras. Therefore, $\mathbb{T}_{\Sigma/E}(X)$ does satisfy the equations E. We just need to prove the inclusion $Thm(\mathbb{T}_{\Sigma/E}(X)) \subseteq PThm(E)$. Suppose $\mathbb{T}_{\Sigma/E}(X) \models u = v$. This means that for each assignment $a \in [X \to T_{\Sigma/E}(X)]$ we have $ua = va$. But for any such a we can find a substitution $\theta : X \to T_\Sigma(X)$ such that for each $x \in X$ we have $a(x) = [x\theta]_E$. That is, any $a \in [X \to T_{\Sigma/E}(X)]$ can be expressed as a composition $a = \theta; [\_]_E$, where $[\_]_E : \mathbb{T}_{\Sigma(X)} \to \mathbb{T}_{\Sigma(X)/E} : t \mapsto [t]_E$ is the unique $\Sigma(X)$-homomorphism[2] from the $\Sigma(X)$-term algebra to the initial $(\Sigma(X), E)$-algebra. But by the Freeness Corollary we then have:

$$\_a = \_\theta; [\_]_E$$

[1] This inclusion, i.e., the Completenes Theorem, only depends on the assumption that Σ is sensible. The other assumptions allow a simpler proof and are added for that reason.

[2] That the mapping $t \mapsto [t]_E$ is indeed a $\Sigma(X)$-homomorphism from $\mathbb{T}_{\Sigma(X)}$ to $\mathbb{T}_{\Sigma(X)/E}$ follows easily from the definition of $\mathbb{T}_{\Sigma(X)/E}$. Uniqueness then follows from the initiality of $\mathbb{T}_{\Sigma(X)}$.

In particular this holds for the assignment $b = \eta_X; [\_]_E$, where, as usual, η_X denotes the identity substitution. Therefore, we have:

$$ub = [u\,\eta_X]_E = [u]_E = [v]_E = [v\,\eta_X]_E = vb$$

But this exactly means that $u =_E v$ and therefore that $u = v \in PThm(E)$. q.e.d.