Equivalent Models for CFLs

Note: "Model2s can simulate Model1s" is shorthand for "For any Model1 A, there is a Model2 B such that $L(A) = L(B)$." For example, DFAs, NFAs, and regular expressions can all simulate each other; PDAs can simulate DFAs but not vice versa.

You do not need to justify your answers today.

1. CFG $\rightarrow$ PDA

In this problem you will show that PDAs can simulate CFGs.

a. For convenience we first define a GPDA to be like a PDA except that it can push multiple symbols onto the stack during a single transition. Show how PDAs can simulate GPDAs.

b. Show how GPDAs can simulate CFGs. *(Hint: You only need 3 states.)*

2. PDA $\rightarrow$ CFG

In this problem you will do part of the proof that CFGs can simulate PDAs.

a. Call a PDA *well-behaved* if

- it has only one accept state
- it always empties its stack before accepting
- each transition does exactly one of pushing or popping from the stack - never both and never neither.

Show how well-behaved PDAs can simulate PDAs.

(It remains to show that CFGs can simulate well-behaved PDAs; you can find this proof in the textbook if you are interested.)

3. CFG $\to$ Chomsky Normal Form

(Sipser Definition 2.8) A context-free grammar is in *Chomsky normal form* (CNF) if every rule is of the form

$$A \to BC$$
$$\text{or}$$
$$A \to a$$

where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

We will show that CNF CFGs can simulate arbitrary CFGs, by converting an arbitrary CFG to satisfy each of these criteria in order:

1. The start variable does not appear on the right-hand side of any rule.
2. There are no $A \to \varepsilon$ rules (other than potentially one for the start variable).
3. There are no $A \to B$ rules (for B a variable).
4. There are no $A \to w$ rules for $|w| > 2$.
5. For all $A \to XY$ rules, X and Y are variables.

I've listed the parts below in the natural order for the proof, but it may make sense to skip (b) and (c) at first since they're the hardest so the others can be used as warm-up.

a. Given an arbitrary CFG, show how to create an equivalent one whose start variable does not appear on the right-hand side of any rule. (i.e. it follows criterion 1 above).

b. Given a CFG satisfying criterion 1, show how to create an equivalent one satisfying 1 and 2 (i.e. show how to remove $A \to \varepsilon$ rules on everything except the start variable).

c. Given a CFG satisfying criteria 1 and 2, show how to create an equivalent one satisfying 1-3 (i.e. remove $A \to B$ rules).

d. Given a CFG satisfying criteria 1-3, show how to create an equivalent one satisfying 1-4 (i.e. remove rules with long right-hand sides).

e. Given a CFG satisfying criteria 1-4, show how to create an equivalent one satisfying 1-5 (i.e. $A \to XY$ rules must not contain terminals).