Final Exam Additional Practice

a Recall that the pumping lemma for regular languages says the following: If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

(a) for each $i \geq 0, xy^i z \in A$,

(b) $|y| > 0$, and

(c) $|xy| \leq p$.

Use the pumping lemma to prove $L = \{0^n 1^m \mid n < m\}$ is nonregular. Additionally, the string s you choose in your proof must have at least 4 times as many 1s as 0s.

b Call an n-state NFA *complex* if every DFA equivalent to it has at least 2^n states. Briefly justify that a complex NFA cannot have any non-self-loop ε-transitions, then construct a complex 2-state NFA.

c Suppose that the language of PDA M is non-regular. What is the tightest possible lower bound you can give for the number of states in M? Similarly, if the language of CFG C is nonregular, what is the tightest lower bound for the number of rules in C?

d A 2-head TM is like a normal TM except that it has two (distinguishable) tape heads; at every computation step, it reads two symbols, writes two symbols, and then moves each head left, right, or keeps it where it is. The two heads both begin at the first tape cell. Briefly describe (at a high level) how to simulate a 2-head TM M using a normal TM R. If the tape alphabet of M is Γ_M, formally specify the tape alphabet Γ_R used by your constructed R.

e Describe two operations where the class of (Turing-)recognizable languages is not closed under either of those operations, but the class of (Turing-)decidable languages is. Can there also exist operations for which the class of decidable languages is not closed but the class of recognizable languages is?

f Show that $INF_{enum} = \{\langle M \rangle \mid M \text{ is an enumerator and } |L(M)| \text{ is infinite}\}$ is undecidable.

g Let $\Gamma = \{0, 1, \text{␣}\}$ be the tape alphabet for all TMs in this problem. Define the busy beaver function $BB : \mathbb{N} \to \mathbb{N}$ as follows. For each value of k, consider all k-state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that BB is not a computable function.

h Prove that $\{\#^n \mid n \text{ is prime}\} \in P$. Would your same solution also show that $PRIMES = \{\langle n \rangle \mid n \text{ is prime}\} \in P$? Do you think $PRIMES$ is in P (on this last bit, no justification required and spend no more than a minute)?