Hierarchy Theorems and Relativization

Hierarchy Theorems

I am going to elide some details in the theorems below by informally defining "nice" functions to be ones that are "efficiently computable"; all functions we regularly talk about are nice, including logs, polynomials, exponentials, and sums/products thereof, as long as all numeric constants used are rationals (e.g. $n^{3.1}$ is nice but n^{π} is not).

These theorems use small-o notation; all you need to know about it is that small-o is to "less than" as big-O is to "less than or equal to", e.g. both $n^{1.99}$ and n^2 are $O(3n^2)$, but $n^{1.99}$ is $o(3n^2)$ while n^2 is *not* $o(3n^2)$ (as with big-O, constant factors are effectively ignored). Formally, $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

(Sipser Theorem 9.3, "Space hierarchy theorem") For any nice function $f : \mathbb{N} \to \mathbb{N}$, a language A exists that is decidable in $O(f(n))$ space but not in $o(f(n))$ space.

(Sipser Theorem 9.10, "Time hierarchy theorem") For any nice function $t : \mathbb{N} \to \mathbb{N}$, a language A exists that is decidable in $O(t(n))$ time but not decidable in time $o(t(n)/\log t(n))$.

a Prove that $NL \subsetneq PSPACE$. *Hint: recall that for $f(n) \geq \log(n)$, $NSPACE(f(n)) \subseteq SPACE(f^2(n))$ (Savitch's Theorem)*

b Prove that for any two *reals* $0 \leq a < b$, $SPACE(n^a) \subsetneq SPACE(n^b)$.

c Describe the error in the following fallacious "proof" that $P \neq NP$. Assume towards contradiction that $P = NP$. Then $SAT \in P$ and so for some k, $SAT \in TIME(n^k)$. Because every language in NP is polynomial time reducible to SAT, you have $NP \subseteq TIME(n^k)$. Therefore, $P \subseteq TIME(n^k)$. But by the time hierarchy theorem, $TIME(n^{k+1})$ contains a language that isn't in $TIME(n^k)$, so we have $P \subseteq TIME(n^k) \subsetneq TIME(n^{k+1}) \subseteq P$, contradiction.

d Consider the major complexity classes we've discussed this semester: $L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq NPSPACE \subseteq EXPTIME$. (Make sure you could justify each of those subset relationships.) What additional relationships are known to hold between those classes - which classes are known to be equal, and which classes (not necessarily adjacent in the list) are known to be *strict* subsets of each other?

Relativization

Let P^A be the class of languages decidable with a polynomial time oracle Turing machine that uses oracle A. Define the class NP^A similarly for oracle NTMs.

e Explain why $NP \subseteq P^{SAT}$ and $coNP \subseteq P^{SAT}$.

f Consider a proof that attempts to show $P = NP$ using the following template:

> Let f be the poly-time-computable function __________, and let L be in NP. Then there is a poly-time verifier V for L. Now we construct $M =$ "On input w, simulate V on $f(w)$ and *accept/reject* exactly as it does." We see that M is a poly-time decider for L, so $L \in P$. Thus $P = NP$.

Show how to modify any proof using this template into a proof that for every K, $P^K = NP^K$.

g A proof which can be trivially modified in the manner from the previous part is said to be a *relativizing* proof. So a relativizing proof that $P \neq NP$ could be modified to show that for all K, $P^K \neq NP^K$. However, show that in reality $P^{TQBF} = NP^{TQBF}$ (and so we can conclude that no relativizing proof of $P \neq NP$ can ever be correct). *(Hint: show $P^{TQBF} \subseteq NP^{TQBF} \subseteq NPSPACE \subseteq PSPACE \subseteq P^{TQBF}$)*