Space Complexity

(from Sipser Definition 8.1) Let M be a (deterministic) TM that halts on all inputs. The space complexity of M is the function $f: \mathbb{N} \to \mathbb{N}$, where f(n) is the maximum number of tape cells that M scans on any input of length n.

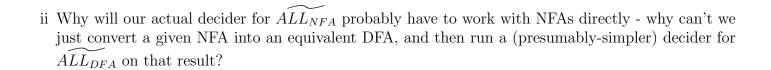
If N is an NTM wherein all branches halt on all inputs, we define its space complexity f(n) to be the maximum number of tape cells that N scans on any branch of its computation for any input of length n. (from Sipser Definition 8.2) Let $f: \mathbb{N} \to \mathbb{R}^+$ be a function.

 $SPACE(f(n)) = \{L \mid L \text{ is a language decided by an } O(f(n)) \text{ space (deterministic) TM}\}.$ $NSPACE(f(n)) = \{L \mid L \text{ is a language decided by an } O(f(n)) \text{ space NTM}\}.$

a Recall that $SAT = \{ \langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula} \}$. Show that $SAT \in SPACE(n)$ (even though SAT is NP-complete!).

b Let $\widetilde{ALL}_{NFA} = \{\langle A \rangle \mid A \text{ is an NFA and } L(A) \neq \Sigma^* \}$. We will show that $\widetilde{ALL}_{NFA} \in NSPACE(n)$. $(\widetilde{ALL}_{NFA} \text{ is what I would previously have called } co-ALL_{NFA}, \text{ but this notation was confusing since there are also complexity classes called things like coNP.)}$

i Assume that some NFA $N=(Q,\Sigma,\delta,q_0,F)$ rejects at least one string, and let w be some shortest rejected string. Find an upper bound on the length of w in terms of some features of N. (Hint: first convert the NFA to a DFA)



iii Construct a linear-space NTM deciding $\widetilde{ALL_{NFA}}$. Hint: the rough strategy is to nondeterministically guess a string of length \leq your bound from part i, and accept if that string is rejected. But notice that you can't actually store the whole string.

- c We will now show that for $f(n) \ge n$, $NSPACE(f(n)) \subseteq SPACE(f^2(n))$.

ii Construct a space-efficient decider for $TIMED-YIELDS_N$.