Computability Exam Additional Practice

1. Oracles

Let $B = \{\langle M^{\square} \rangle \mid \text{for every language } K \neq \Sigma^*, K \subsetneq L(M^{\boxed{K}})\}$.

a Design an oracle TM $R^{\square}$ such that $\langle R^{\square} \rangle \in B$ and for some w and K, $R^{\boxed{K}}$ rejects w.

b Prove that B is undecidable.

2. Computation Histories (The Tiling Problem)

Imagine we have a finite set $\{t_1, \cdots t_n\}$. Each t_i is a colorful square *tile* that cannot be rotated. We only allow two tiles to be placed next to each other if that pairing looks nice, as represented by two relations: xRy iff y is allowed to be placed directly to the Right of x, and xBy iff y is allowed to be placed directly Below x. A *tilekit* is a (finite) set of tiles along with the two relations. We will consider variations of the following question: given a tilekit, is it possible, if you have infinitely many copies of each tile, to tile the lower-right quadrant of the plane? i.e. your tiling must have an upper edge and a left-hand edge, but then fill all space forever going downwards and rightwards. Or more precisely, for *every* n, you can tile the upper-left $n \times n$ region of the quadrant. For example, if our tilekit contains a single tile t where tRt and tBt, then we can tile the quadrant; if there is a single tile t and we *don't* have both tRt and tBt, then we can't tile the quadrant.

a Let $TILING = \{\langle X \rangle \mid \text{tilekit } X \text{ can tile the quadrant}\}$. Prove that *co*–$TILING$ is recognizable.

b Let $SEEDED{-}TILING = \{\langle X, S \rangle \mid \text{tilekit } X \text{ can tile the quadrant using } S \text{ as the upper-left tile}\}$. We will use computation histories to prove that $SEEDED - TILING$ is undecidable.

Fix TM $M = (Q, \Sigma, \cdots)$. Our strategy will be to have each row in a tiling represent a configuration of a TM M, and for a given row, the only possible next row will be the next configuration in a valid computation history. Each tile will be defined by three symbols from $Q \cup \Gamma$ and an optional left or right arrow. Either the tile will have exactly one state symbol, or it will have zero and it will have an arrow pointing in the direction where the tape head can be found. For example, the configuration $012q0$ would be represented by $0\overset{\rightarrow}{1}2$, $12q$, $2q0$, $q0␣$, $0\overset{\leftarrow}{␣}␣$, and then an infinite sequence of $␣\overset{\leftarrow}{␣}␣$ tiles.

Design relations R and B such that, *if* some row of a tiling represents a valid configuration, there will be at most one possible row below it in the tiling and that row will represent the next configuration in a valid computation history of M. *(You do not need to work out every last formal detail of this construction - feel free to describe parts of it at a high level, and to skip annoying edge cases as long as they seem probably solvable using similar methods.)*

c Choose a tile S (adding new tiles and updating R and B as necessary) such that any infinite tiling seeded by tile S must contain some row (not necessarily the first row) representing a valid start configuration for M on input ε.

d Prove that $SEEDED-TILING$ is undecidable.