Turing-reducibility and Oracle Machines

a i. Fix TM M, and construct TM N as follows:

$N =$ "On any input:

- Run M in parallel on all strings in Σ^* *(using our usual strategy of gradually simultaneously increasing both the number of strings being tested and the number of timesteps to test each string for.)*
- If M accepts any of these strings, *accept*."

What is $L(N)$?

$$L(N) = \begin{cases} \underline{\hspace{6cm}} & \text{if } \underline{\hspace{6cm}} \\ \underline{\hspace{6cm}} & \text{if } \underline{\hspace{6cm}} \end{cases}$$

ii. An oracle TM M^L is a regular TM that can also as a single step ask an "oracle" whether a given string is in the language L. Consider this oracle TM:

$T^{A_{TM}} =$ "On input $\langle M \rangle$, where M is a TM:

- Use M to construct N as described in part i.
- Query the oracle about $\langle N, \varepsilon \rangle$ (i.e. ask it to determine whether $\langle N, \varepsilon \rangle \in A_{TM}$).
- If the oracle answers NO, *accept*; if YES, *reject*."

What language L does $T^{A_{TM}}$ decide? (We will then say that L is decidable *relative to* A_{TM}, or that L is *Turing reducible* to A_{TM}, written as $L \leq_T A_{TM}$).

b A_{TM} is decidable relative to A_{TM}, since we can build a machine $R^{A_{TM}}$ which simply queries its oracle about the input string and accepts/rejects if it gets a yes/no answer. Where specifically is the flaw in the following proof that A_{TM} is *undecidable* relative to A_{TM}?

"Proof": Assume towards contradiction that A_{TM} is decidable relative to A_{TM}. Then there is an oracle machine $H^{A_{TM}}$ which decides A_{TM}. Now define TM D as follows:

$D =$ "On input $\langle M \rangle$, where M is a TM:

- Simulate $H^{A_{TM}}$ on input $\langle M, \langle M \rangle \rangle$.
- Output the opposite of what $H^{A_{TM}}$ outputs. That is, if $H^{A_{TM}}$ accepts, *reject*; and if $H^{A_{TM}}$ rejects, *accept*."

Now consider what happens when we run D on $\langle D \rangle$, and we reach a contradiction exactly as in the original proof that A_{TM} is undecidable (Sipser p207).

c Show that if $A \leq_T B$ and $B \leq_T C$, then $A \leq_T C$. *(Show that if some R^B decides A and some S^C decides B, then you can construct a T^C that decides A. Note that your T^C does not have access to an oracle for B.)*

d Show that for any two languages A and B, a language J exists, where $A \leq_T J$ and $B \leq_T J$.

e Let $A'_{TM} = \{\langle M, w\rangle \mid M \text{ is an oracle TM and } M^{A_{TM}} \text{ accepts } w\}$. Prove that A'_{TM} is undecidable relative to A_{TM}. Similarly, argue that $A''_{TM} = \{\langle M, w\rangle \mid M \text{ is an oracle TM and } M^{A'_{TM}} \text{ accepts } w\}$ is undecidable relative to A'_{TM}.

f Prove that A_{TM} is decidable relative to A'_{TM}.