

## Reductions using computation histories

(adapted from Sipser Definition 5.5) Let  $M$  be a Turing machine and  $w$  an input string. A computation history for  $M$  on  $w$  is a finite sequence of configurations,  $C_1, C_2, \dots, C_l$ , where  $C_1$  is the start configuration of  $M$  on  $w$ , and each  $C_i$  legally follows from  $C_{i-1}$  according to the rules of  $M$ . A computation history is *accepting* if  $C_l$  is an accepting configuration, and *rejecting* if  $C_l$  is a rejecting configuration.

Assume that we encode computation history  $H = C_1, C_2, \dots, C_l$  by

$\langle H \rangle = \#C_1\#C_2^R\#C_3\#C_4^R\cdots C_l^{(R)}\#$ , i.e. we reverse every other configuration and intersperse  $\#$  signs around them. Fix TM  $M$  and string  $w$ , and let  $ACCHIST_{M,w} = \{\langle H \rangle \mid H \text{ is an accepting configuration history for } M \text{ on } w\}$ . We are going to do parts of the construction for a PDA which recognizes the language  $\overline{ACCHIST_{M,w}}$ . Describe all PDAs at a high level (you don't need to actually draw state diagrams).

- a Describe a PDA which accepts if a given string  $x$  is *not* a valid configuration of  $M$ . (Don't worry about whether  $x$  is a *reachable* configuration; just check whether it has the correct form to be a configuration at all.)
  
  
  
  
  
  
  
  
  
  
- b (For this and all remaining parts of the construction, you may assume that the input  $\langle H \rangle$  does have at least two  $\#$  symbols, and that between each pair of  $\#$  symbols is some valid configuration of  $M$ .) Describe a PDA which accepts input  $\langle H \rangle$  if  $H$  does *not* start with a valid  $C_1$  (for our fixed  $M$  and  $w$ ).

- c Describe a PDA which accepts if input  $\langle H \rangle$  does *not* end with a valid accepting configuration  $C_l$ .
- d Assume that we have some way to finish this construction, i.e. given a TM  $M$  and a string  $w$  we can create a PDA  $P$  which recognizes  $\overline{ACCHIST_{M,w}}$ . Show that  $ALL_{PDA} = \{\langle M \rangle \mid M \text{ is a PDA and } L(M) = \Sigma^*\}$ , and the similarly-defined  $ALL_{CFG}$ , are undecidable languages.

- e Let's go back and fill in more details of the PDA's construction. Assume for this part that  $H = C_1, C_2$ , and that the final symbols of  $C_1$  and  $C_2$  are the same. Describe a PDA which accepts input  $\langle H \rangle$  if  $C_2$  is *not* the valid next configuration for  $M$  starting from  $C_1$  (do not assume that  $C_1$  is the starting configuration).

f Extending part e, describe a PDA which accepts input  $\langle H \rangle$  if *some*  $C_i$  is not the valid next configuration for  $M$  starting from  $C_{i-1}$ . Then once we have all the pieces of the construction, how do we tie them together into a single PDA which recognizes  $\overline{ACCHIST_{M,w}}$ ?

g If we tried to adapt our construction to recognize  $ACCHIST_{M,w}$  instead, what parts could be easily adapted and what parts could not?