Reductions and Rice's Theorem

a. We will prove that E_{TM} is undecidable via reduction from A_{TM}.

i. Fix TM M and string w. Consider the following TM $T_{M,w}$:

$T_{M,w}$ = "On input x:

- If $x \neq w$, *reject.*
- If $x = w$, run M on input w and accept if M does."

What is $L(T_{M,w})$?

$$L(T_{M,w}) = \begin{cases} \underline{\hspace{4cm}} & \text{if } \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} & \text{if } \underline{\hspace{4cm}} \end{cases}$$

ii. Assume towards contradiction that TM R decides E_{TM} and construct TM S that decides A_{TM} as follows. S = "On input $\langle M, w \rangle$, where M is a TM and w is a string:

- Use the description of M and w to construct the TM $T_{M,w}$ described in part i.
- Run R on input $\langle T_{M,w} \rangle$.
- If R accepts, ____________________; if R rejects, ____________________."

Fill in the blanks above and argue that the resulting TM S is a decider for A_{TM}. (Recall that $A_{TM} = \{\langle M, w \rangle \mid \text{TM } M \text{ accepts } w\}$ is undecidable, so this is our contradiction.)

b. Prove that $CF_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is context-free}\}$ is undecidable via reduction from A_{TM}.

c. Fix X to be some set of ten recognizable languages, none of which are $\emptyset$. Prove that $L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \in X\}$ is undecidable via reduction from A_{TM}.

d. Consider this incomplete theorem:

Fix X to be some set of recognizable languages such that __________________. Then $L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \in X\}$ is undecidable.

In the previous problem we've seen one way of filling in the blank: "$|X| = 10$, $\emptyset \notin X$". However, it is possible to significantly relax these requirements in order to make the theorem far more powerful, i.e. applicable to many more possible values of X. Come up with the most powerful version of this theorem you can, and prove that your theorem is true. Prove that you have found the most powerful possible version of the theorem. *(Hint: Instead of trying to find the most powerful version all at once, you may want to relax one requirement at a time, tweak the proof to show your new version is correct, and then think of some X that is not yet covered by your predicate and consider whether the predicate must be relaxed further to include that X, or if the corresponding language is indeed decidable.)*

e. The theorem from the previous part is (in a slightly different form) known as Rice's Theorem. Discuss its implications for real-world problems we might want to solve - e.g. what limitations does the theorem imply on all attempts to write a virus scanner, and what can we do to work around those limitations?