Enumerators

Note: "Shortlex" (or just "standard") order for strings is shorter strings before longer ones, and after that by 'dictionary' order, e.g. for $\{0,1\}^*$ it's $[\varepsilon, 0, 1, 00, 01, 10, 11, 000, \cdots]$

a. Let Turing Machine M recognize language L.

   i (Sipser 3.6) Why might the following enumerator not enumerate L?
   E = "Ignore the input.

   - For each string s of Σ^* (in standard order):
     - Run M on s. If it accepts, *print* s."

   ii Design an enumerator that enumerates L.

b. Prove that a language is (Turing-)recognizable iff some enumerator enumerates it. (And as a result, we will sometimes use the name "RE", i.e. "recursively *enumerable*", to refer to the class of recognizable languages.)

c. (Sipser 3.18) Show that a language is decidable iff some enumerator enumerates the language in the standard string order.

d. (Sipser 3.19) Show that every infinite Turing-recognizable language has an infinite decidable subset. *(Hint: use the result from the previous problem.)*

e. (Sipser 4.30) Let A be a Turing-recognizable language consisting of descriptions of Turing machines, $\{\langle M_1 \rangle, \langle M_2 \rangle, \cdots\}$, where every M_i is a decider. Prove that some decidable language D is not decided by any decider M_i whose description appears in A. (Hint: You may find it helpful to consider an enumerator for A.)

f. (Sipser 4.20) Let A and B be two disjoint languages. Say that language C separates A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-Turing-recognizable languages are separable by some decidable language. *(A language is co-RE if its complement is RE. Hint: First show that* every *string is in at least one of* $\overline{A}$ *and* $\overline{B}$*.)*