$$K \leq_m MP, \quad MP \leq_m HP, \quad K \leq \text{EMPTY}, \quad \neg K \leq_{m FIN}, \quad \neg K \leq_{m \overline{FIN}}$$
Lecture 12: Oracle Turing Machines and Arithmetic Hierarchy

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**Oracle Turing Machine** is a 2-tape Turing machine $M = (Q, \Sigma, \Gamma, \delta, \ldots, s, t, r)$. The first tape is the regular tape of the Turing machine which initially holds the input, and the machine can read/write from. The second tape is called the query tape. The Oracle Turing machine also has 3 special states — the query state $q_?$, and the answer states $q_{\text{yes}}$ and $q_{\text{no}}$.

An oracle Turing machine is executed with a language $B$; $B$ is said to be the oracle. The machine $M$ executed with oracle $B$ is denoted as $M^B$. Such a machine proceeds like a normal (2-tape) Turing machine, until it reaches the query state $q_?$. Let $x$ be the string written on its query tape at this point. In the next step, $M^B$ moves to state $q_{\text{yes}}$ if $x \in B$, and to state $q_{\text{no}}$ if $x \notin B$. The computation of $M^B$ then proceeds as normal, until the next point in time when it reaches the query state $q_?$. This process keeps repeating until the machine halts in $t$ or $r$.

**Definition 1.** Given a language $B$, 

\[ A \text{ is } \text{r.e. (recursively enumerable) in } B \text{ if } A = L(M^B). \]

- $A$ is recursively enumerable in $B$ if there is an oracle Turing machine $M$ such that $A = L(M^B)$.
- $A$ is recursive in $B$ if there is a total oracle Turing machine $M$ such that $A = L(M^B)$. When $A$ is recursive in $B$, we also say that $A$ Turing reduces to $B$, and is denoted as $A \leq_T B$.

**Proposition 1.** If $A \leq_T B$ and $B \leq_T C$ then $A \leq_T C$.

**Goal:** Design total TM $M$ s.t. $A = L(M^B)$.

$\Rightarrow$: Run $M$, and whenever $M$ asks a query, it will run $M^B$ to answer the query.

**Proposition 2.** If $A \leq_T B$ then $\overline{A} \leq_T B$.

- $\exists G$ $A$ is recursive then $\overline{A}$ is recursive
- $\exists$ total TM $M$ s.t. $A = L(M^B)$

To test $\overline{A}$: Run $M$ with oracle $B$ and flip answer.

**Proposition 3.** $A \leq_T B$ iff $A$ and $\overline{A}$ are both recursively enumerable in $B$.

($\Rightarrow$) If $\exists$ total TM $M$ s.t. $A = L(M^B)$, then $A$ is r.e. in $B$.

($\leq$) Let $M_1$ and $M_2$ be TM s.t. $A = L(M_1^B)$ and $\overline{A} = L(M_2^B)$

New Algo: Run $M_1$ & $M_2$ "in parallel".

Due prop. 2.
Arithmetic Hierarchy is a hierarchy of classes inductively defined as follows.

\[
\begin{align*}
\Sigma^0_n & = \text{RE} \\
\Delta^0_n & = \text{REC} \\
\Sigma^{0+1}_n & = \{ L(M^B) \mid B \in \Sigma^0_n \} \\
\Delta^{0+1}_n & = \{ L(M^B) \mid B \in \Sigma^0_n, \text{M}^B \text{ is total} \} = \{ A \mid \exists A \in \Sigma^0_n, \exists B \in \Delta^0_n, A \leq_T B \} \\
\Pi^0_n & = \{ A \mid A \in \Sigma^0_n \}
\end{align*}
\]

Observe that \( \Pi^0_1 = \text{co-RE} \).

**Proposition 4.** Prove that for all \( n \geq 1 \), \( \Delta^0_n = \Sigma^0_n \cap \Pi^0_n \).

\[
\Delta^0_n \subseteq \Sigma^0_n \cap \Pi^0_n : A \in \Delta^0_n, \exists B \in \Sigma^0_{n-1}, A \leq_T B
\]

\[
A \text{ is re in } B \text{ and } \overline{A} \text{ is r.e in } B \Rightarrow A \in \Sigma^0_n, \overline{A} \in \Sigma^0_n.
\]

\[
\Sigma^0_n \cap \Pi^0_n \subseteq \Delta^0_n : A \in \Sigma^0_n \cap \Pi^0_n, \exists \text{TM } M_1, M_2, B_1 \text{ and } B_1, B_2 \in \Sigma^0_{n-1} \text{ s.t.} \\
A = L(M^B_1) \text{ and } \overline{A} = L(M^B_2) \\
B = \exists x \mid x \in B_1 B_2 \cup \exists x \mid x \in B_2 B_1 \in \Sigma^0_{n-1} \text{ w.r.t. } B
\]

"Run \( M_1 \) & \( M_2 \) in parallel with \( B \)."

**Proposition 5.** Prove that \( A \in \text{RE} \) iff there is a recursive relation \( R \) such that \( A = \{ x \mid \exists y. R(x, y) \} \).

\((\Rightarrow)\) \( A \in \text{RE} \). \( \exists \text{ TM } M \text{ s.t. } A = L(M) \\
R_m = \exists x \mid x \in M \text{ accepts } x \text{ within } t \text{ steps} \)

\( R_m \text{ recursive: "Run } M \text{ on } x \text{ for } t \text{ steps"} \)

\( A = \exists x \mid x \text{ is accepted by } M \\
A = \exists x \mid \exists y. R(x, y) \)

\((\Leftarrow)\) \( R \text{ is recursive } \Rightarrow \exists \text{ TM } M \text{ s.t. } R = L(M) \\
A = \exists x \mid \exists y. R(x, y) \\
\text{Alg for } \overline{A} \text{ input } x. \text{ Run } M \text{ on } (x, y) \)

**Theorem 6.**

1. A set \( A \in \Sigma^0_n \) iff there is a recursive relation \( R \) such that

\[
A = \{ x \mid \exists y_1 \forall y_2 \exists y_3 \cdots Q y_n R(x, y_1, y_2, \ldots y_n) \}
\]

where \( Q = \exists \) if \( n \) is odd, and \( Q = \forall \) if \( n \) is even.

2. A set \( A \in \Pi^0_n \) iff there is a recursive relation \( R \) such that

\[
A = \{ x \mid \forall y_1 \exists y_2 \forall y_3 \cdots Q y_n R(x, y_1, y_2, \ldots y_n) \}
\]

where \( Q = \forall \) if \( n \) is odd, and \( Q = \exists \) if \( n \) is even.
Problem 1. Prove that $\text{EMPTY} = \{ \langle M \rangle \mid L(M) = \emptyset \} \in \Pi^0_1$. 

Need a recursive relation $R \subseteq t$ such that $\text{EMPTY} = \{ x \mid \forall y \exists t \ R(x, y, t) \}$ 

$x \in \text{EMPTY} \iff \forall y \exists t \ R(x, y, t)$ 

Problem 2. Prove that $\text{TOTAL} = \{ \langle M \rangle \mid M \text{ is total} \} \in \Pi^0_2$. 

Need a recursive relation $R \subseteq t$. $\text{TOTAL} = \{ x \mid \forall y \exists z \ R(x, y, z) \}$ 

Take $R = \{ x, y, z \mid M_x \text{ halts in } z \text{ steps on input } y \}$ 

$\text{TOTAL} = \{ x \mid \forall y \exists z \ R(x, y, z) \}$ 

Problem 3. Prove that $\text{FIN} = \{ \langle M \rangle \mid L(M) \text{ is finite} \} \in \Sigma^0_2$. 

Problem 4. Prove that $\text{COF} = \{ \langle M \rangle \mid L(M) \text{ is cofinite} \} \in \Sigma^0_3$. 

Theorem 7. The arithmetic hierarchy is strict. That is, for every $n$, (a) $\Sigma^0_n \cap \Pi^0_n = \emptyset$ and $\Pi^0_n \subseteq \Delta^0_{n+1}$. 

3
Definition 2. A language $A$ is $C$-hard, for a class $C$ of languages, with respect to $\leq_m$ if for every $B \in C$, $B \leq_m A$. And $A$ is $C$-complete if $A \in C$ and $A$ is $C$-hard.

Proposition 8. 1. $\text{EMPTY}$ is $\Pi^0_1$-complete.

2. $\text{TOTAL}$ is $\Pi^0_2$-complete.

3. $\text{FIN}$ is $\Sigma^0_2$-complete.

4. $\text{COF}$ is $\Sigma^0_3$-complete.