Finite Model Theory
Classical Model Theory

Study of mathematical objects (graphs, algebraic structures) through non-logic.

Gödel's Completeness Theorem If \( \mathcal{T} \) is recursively enumerable and \( \varphi \) is a sentence then the problem of determining if \( \mathcal{T} \models \varphi \) is recursively enumerable.

- The set of valid sentences is recursively enumerable.

Compactness Theorem A set \( \mathcal{T} \) of sentences \( \mathcal{T} \) is unsatisfiable if \( \exists \text{ finite } \mathcal{T}_0 \subseteq \mathcal{T} \) such that \( \mathcal{T}_0 \) is unsatisfiable.

Proof By Skolemization \( \mathcal{T} \) is a set universally quantified sentences.

\[ \mathcal{T}^* = \{ \forall \mathcal{V} \left[ \forall x_1 \rightarrow t(x_1, \ldots, \forall x_n) \varphi \in \mathcal{T} \right] \}
\]

and \( t, \ldots, t_n \) are ground terms.

\( \mathcal{T} \) is satisfiable iff \( \mathcal{T}^* \) is satisfiable.
$T^*$ unsatisfiable $\implies \exists$ finite $\Delta \subseteq T^*$

that is unsatisfiable.

$T^* = \exists \varphi \in T^*$ \text{ some ground instantiation } \varphi \text{ of } \varphi \in \Delta \exists

$T^*$ is unsatisfiable.

**Example**

There sentences $\varphi$ such that $\forall A \; A \not\models \varphi \implies A$ is finite.

$\varphi = \exists x \neq y x = y$

**Proposition** There is no sentence $\varphi$ such that (a) every structure satisfying $\varphi$ is finite, and (b) $\varphi$ has models of arbitrary size.

**Proof** Assume $\varphi$ has finite models of size $\eta_k = \exists x_1 \exists x_2 \ldots \exists x_k \wedge \forall i \forall j \; (x_i = x_j)$

If $A \not\models \eta_k$ then $\nu(A)$ has at least $k$ elements.

$T^* = \exists \varphi \exists \varphi_2 \ldots \exists \eta_k \exists k \subseteq T^*$

Since finite subset of $T^*$ is satisfiable.
If \( \mathcal{T} \) is a countable signature and \( \mathcal{A} \) is a set of \( \mathcal{T} \)-sentences that satisfiable then there is a countable structure \( A \) such that \( A \models \mathcal{T} \).

**Proposition** There are non-isomorphic structures \( A \) and \( B \) such that 
\[ \text{Th} (A) = \text{Th} (B) \]

**Proof** 
\[ \text{Th} (\mathbb{R}, <) = \text{Th} (\mathbb{Q}, <) \]
\[ \text{Th} (\mathbb{R}, 0, 1, +, <) = \text{Th} (\mathbb{Q}, 0, 1, +, <) \]

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"Finite Model Theory" Study of first order logic restricted to finite structures

\( \varphi \models_{\mathcal{I}} \) is satisfiable in finite models
- \( \phi | \phi' \) is valid if \( \phi' \) holds in all finite models.

**Trakhtenbrot's Theorem** The problem of checking if a sentence \( \phi \) is true in all finite structures is coRE-complete.

**Proof** \( \text{Fin Valid} \in \text{coRE} \)
- To check if \( \phi \) is satisfiable in a finite model: Enumerate all finite models \( \mathcal{A} \) and check if \( \mathcal{A} \models \phi \).

**Church-Turing Theorem** Validity is RE hard.

\[
\begin{align*}
\text{Input:} & \quad \exists x \forall y \quad S(x, y) \\
\text{Output:} & \quad \forall x \forall y \forall z \quad S(x, z) \land S(y, z) \rightarrow x = y \\
& \quad \forall x \forall S(x, 0)
\end{align*}
\]

MP \( \leq_m \) Validity.
\[ \phi_m = (\phi_{\text{not}} \land \phi_{\text{initial}} \land \phi_{\text{const}}) \implies \phi_{\text{accept}} \]

Can’t use these ideas to
\[ \overline{MP} \leq \text{Validity} \]

Gödel’s Incompleteness Theorem
\[ \text{Th} \left( \mathbb{N}, 0, 1, +, \times, < \right) \text{ is not R.E.} \]
\[ \overline{HP} \leq \text{Th} \left( \mathbb{N}, 0, 1, +, \times, < \right) \]
\[ \phi_{\langle m, n \rangle} = \phi_{\text{initial}} \land \phi_{\text{const}} \implies \phi_{\text{Halt}}. \]

\[ \begin{array}{c}
0 \rightarrow 0 \rightarrow \cdots \\
6
\end{array} \]

Goal: \[ \overline{MP} \leq_m \text{FinValidity}. \]

\[ \forall x \exists S(x, 0) \]
\[ \forall x \forall y \forall z \ S(x, z) \land S(y, z) \implies x = y \]
\[ \forall y \ S(m, y) \]
\[ \forall y \exists z (y = m) \implies \exists n S(y, x) \]

Finite Models

\[ \overline{0} \rightarrow \rightarrow \rightarrow \overline{m} \]

Given \( m \) infinite
\( \Phi_w \) is valid in all finite models iff
Universal TM \( U \) does not accept \( w \).

\[
\Phi_w = \Phi_{\text{finNat}} \land \Phi_{\text{Fin}} \land \Phi_{\text{const}}
\]
\[\rightarrow \quad \exists \text{State}(m, y, \text{acc})
\]

Compactness Theorem does not hold in finite models.

**Proposition**
There is a set of sentences \( \Pi \)
so that every finite subset \( \Pi_0 \subset \Pi \) has
a finite model but \( \Pi \) does not have
any finite model.

**Proof**
\( \Pi = \{ \exists \, \forall \} \land \exists \, \forall \) \( \exists \, \forall \) \( \exists \, \forall \)

**Definition**
Let \( \mathcal{C} \) be some signature.
A homomorphism between \( \mathcal{C} \)-structures 
\( \mathcal{A} \) and \( \mathcal{B} \), 
\( h : u(\mathcal{A}) \rightarrow u(\mathcal{B}) \)
s.t.

- \( \forall c \in \mathcal{C} \), \( h(c^\mathcal{A}) = c^\mathcal{B} \)
- \( \forall a_1, \ldots, a_n \in u(\mathcal{A}) \) \( f \in \mathcal{C} \),
  \( h(f^\mathcal{A}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n)) \)
- \( a_1, \ldots, a_n \in \sigma \) and \( \sigma \subseteq \mathbb{C} \).

\((a_1, \ldots, a_n) \in \mathbb{R}^n \) iff \((h(a_1), \ldots, h(a_n)) \in \mathbb{R}^B\).

An isomorphism between \( A \) and \( B \) is a homomorphism \( h \) that is bijective.

We will say \( A \cong B \) (\( A \) is isomorphic to \( B \)) if \( \exists \) isomorphism \( h \) from \( A \) to \( B \).

**Proposition**: For every finite structure \( A \), \( \exists \) sentence \( \varphi_A \) s.t.

\( \mathcal{M} \models \varphi_A \rightarrow \mathcal{B} \models \varphi_A \rightarrow \mathcal{B} \cong A \).