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Approx Alg'ms

Problem (MAX-SAT)

Given m clauses $C_1, \dots, C_m$ in n vars $x_1, \dots, x_n$,
find an assignment A
maximizing $f(A) = \#$ clauses satisfied by A

e.g. $x_1 \vee x_3$, $\bar{x}_1 \vee x_3$, $\bar{x}_1 \vee \bar{x}_2$, $\bar{x}_3$

$\Rightarrow f(A^*) = 3$

$x_1 = 0$
$x_3 = 1$ or 0

note - NP-hard even if clause length = 2.

Alg'm 1 (Greedy)

polytime

for $i = 1$ to n
  if # occurrences of $x_i \geq$ # occurrences of $\bar{x}_i$
    set x_i true
  else " " false
  remove all clauses containing x_i or $\bar{x}_i$

Analysis: let $m_i = \#$ clauses removed in iteration i

$$\# \text{ clauses satisfied} \geq \sum_{i=1}^{n} \frac{1}{2} m_i = \frac{m}{2} \geq \frac{f(A^*)}{2}$$

$\Rightarrow$ approx factor $\frac{1}{2}$.

Alg'm 2: Randomized

O(n) time

for $i = 1$ to n do $x_i =$ random-bit()

Analysis:

$$E[f(A)] = E\left[\sum_{k=1}^{m} \sigma_k\right] = \sum_{k=1}^{m} E[\sigma_k]$$

$$\sigma_k = \begin{cases} 1 & \text{if } C_k \text{ satisfied} \\ 0 & \text{else} \end{cases}$$

$$E[\sigma_k] = \Pr[C_k \text{ satisfied}]$$

$$= \begin{cases} \frac{1}{2} & \text{if } C_k \text{ has length } 1 \\ \frac{3}{4} & \text{if } C_k \text{ has length } 2 \\ \frac{7}{8} & \text{if } C_k \text{ has length } 3 \\ \vdots \end{cases}$$

$$\geq \frac{1}{2}$$

$$\Rightarrow \quad E[f(A)] \geq \frac{1}{2} m \geq \frac{1}{2} f(A^*)$$

$\Rightarrow$ approx factor $\boxed{\frac{1}{2}}$

Consider special case MAX-1-2-SAT
where all clauses have length 1 or 2

Alg'm 3: Goemans-Williamson technique '93

1. let $y_1, \ldots, y_n \in [0,1]$ be sol'n to
   linear programming relaxation

2. for $i = 1$ to n do
   set $x_i = \begin{cases} 1 & \text{w. prob } y_i \\ 0 & \text{else} \end{cases}$ independently
   "randomized rounding"

Ex $x_1 \vee x_3$, $\bar{x}_1 \vee x_3$, $\bar{x}_1 \vee \bar{x}_2$, $\bar{x}_3$

introduces vars y_1, y_2, y_3, z_1, z_2, z_3, z_4

meaning $z_k = \begin{cases} 1 & \text{if } C_k \text{ sat} \\ 0 & \text{else} \end{cases}$

max $z_1 + z_2 + z_3 + z_4$

s.t. $\quad y_1 + y_3 \geq z_1$

e.g. $\begin{cases} y_1 = 0.5 \\ y_2 = 0 \\ y_3 = 0.5 \end{cases}$

$z_1 = 1$
$z_2 = 1$
$z_3 = 1$
$z_4 = 0.5$
$\Downarrow$
$\underline{3.5}$

$$1 - y_1 + y_3 \geq z_2$$

$$1 - y_1 + 1 - y_2 \geq z_3$$

$$1 - y_3 \geq z_4$$

$$0 \leq y_1, y_2, y_3, z_1, z_2, z_3, z_4 \leq 1.$$

integers

MAX-SAT is equiv to this <u>integer</u> linear prog.

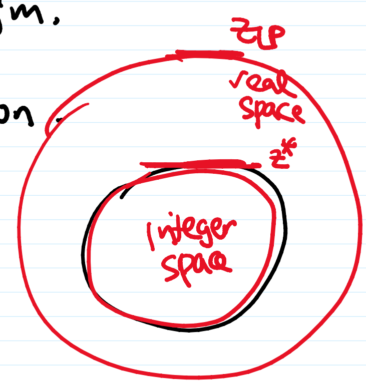
Analysis for MAX-1-2-SAT:

let $z^* = f(A^*)$ opt value.

let A be assignment chosen by alg'm.

let z_{LP} opt value for LP relaxation.

Know $z_{LP} \geq z^*$



$$E[f(A)] = E\left[\sum_{k=1}^{m} \sigma_k\right]$$

$$= \sum_{k=1}^{m} \underline{Pr[C_k \text{ satisfied}]}$$

Case 1. C_k has length 1, say, $\underline{C_k = x_i}$ or $\bar{x}_i$

We have $y_i \geq z_k$.

$Pr[C_k \text{ satisfied}] = y_i \geq z_k$.

Case 2. C_k has length 2, say $C_k = x_i \vee x_j$

We have $\underline{y_i + y_j} \geq z_k$.

$Pr[C_k \text{ satisfied}] = 1 - Pr[C_k \text{ not sat}]$

$$= 1 - (1-y_i)(1-y_j)$$

$$= y_i + y_j - \underline{y_i y_j}$$

$$\geq y_i + y_j - \left(\frac{y_i+y_j}{2}\right)^2$$

(AM $\geq$ GM

$\frac{a+b}{2} \geq \sqrt{ab}$

$(a+b)^2 \geq 4ab$

$(a-b)^2 \geq 0$)

$$\geq z_k - \left(\frac{z_k}{2}\right)^2$$

(since $x - \left(\frac{x}{2}\right)^2$ increas. for $x \in [0,2]$)

$$\geq z_k - \frac{z_k}{4}$$

(since $x^2 \leq x$ for $x \in [0,1]$)

$$= \frac{3}{4} z_k$$

$$\Rightarrow E[f(A)] \geq \sum_{k=1}^{m} \frac{3}{4} z_k$$

$$= \frac{3}{4} z_{LP}$$

$$\geq \frac{3}{4} z^*$$

$\Rightarrow$ expected approx factor $\geq \boxed{\frac{3}{4}}$

General MAX-SAT?

for clause length ℓ, factor $1 - \left(1 - \frac{1}{\ell}\right)^{\ell}$

$$\geq 1 - \frac{1}{e} \approx 0.632$$

Improvement:

w. prob 1/2, run Alg'm 2

" 1/2 Alg'm 3

$\Rightarrow$ expected approx factor $\geq \boxed{\frac{3}{4}}$

(current best alg'm: $\geq \underline{0.7968}$ (2005))