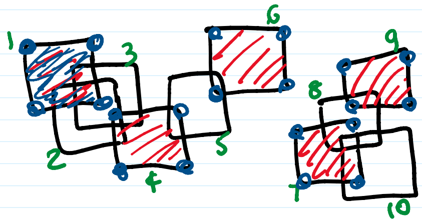
April 30, 2025 12:19 PM

Problem Given a set S of n unit squares in 2D, find a subset $T \subseteq S$ of disjoint squares maximizing $|T|$.

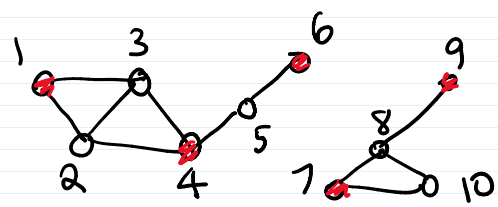


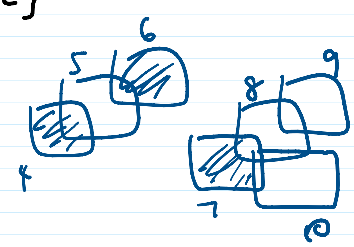
$\{1,5,8\}$

$\{1,4,6,7,9\}$ optimal?

1, 4, 6, 7, 9

let graph $G = (V, E)$,
$V = S$
$E = \{ (s_1, s_2) : s_1 \text{ intersects } s_2 \}$





$\Rightarrow$ max indep set in graph

geom version remains NP-hard $\leftarrow$

approx alg'm?

indep set is very hard to approximate in general
(no factor $n^{1-\epsilon}$ if $P \neq NP$)
polytime

Approx Alg'm 1 (Greedy / Incremental)

$T = \emptyset$
while $S \neq \emptyset$ {
pick any square $t \in S$
insert t to T

remove t and all squares intersecting t from S

}

Runtime: $\leq n$ iteration;
$O(n)$ time per intersection $\Rightarrow$ $O(n^2)$ time

polytime

Analysis of approx factor:

let T^* be opt sol'n.

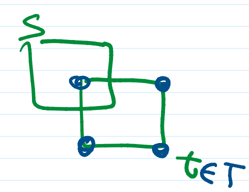
let T be sol'n from our alg'm.

let X = set of all corners of T.

Then $|X| = 4|T|$.

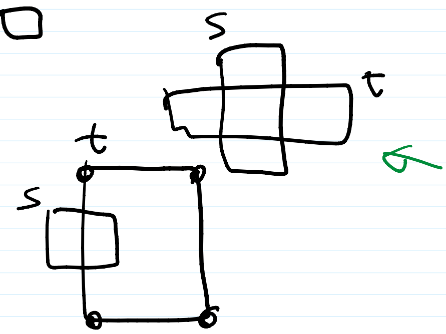
Obs every square $s \in S$ contains at least one pt of X.

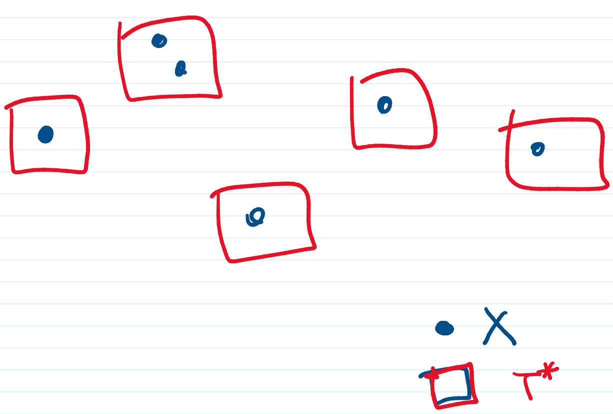
Pf: since s is eventually removed, s must intersect some $t \in T$.



$\Rightarrow$ s must contain a corner of t

$\Rightarrow$ obs. $\square$





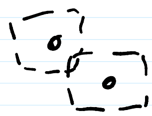
$$|T^*| \leq |X|$$

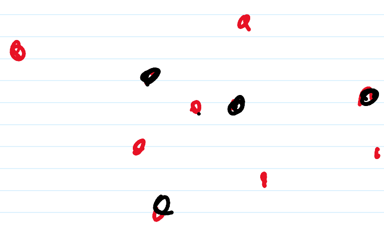
$$|T^*| \le |X| = 4|T| \quad \Rightarrow \quad |T| \ge \frac{|T^*|}{4}$$

$$\Rightarrow \text{ approx factor } \boxed{\tfrac{1}{4}}$$

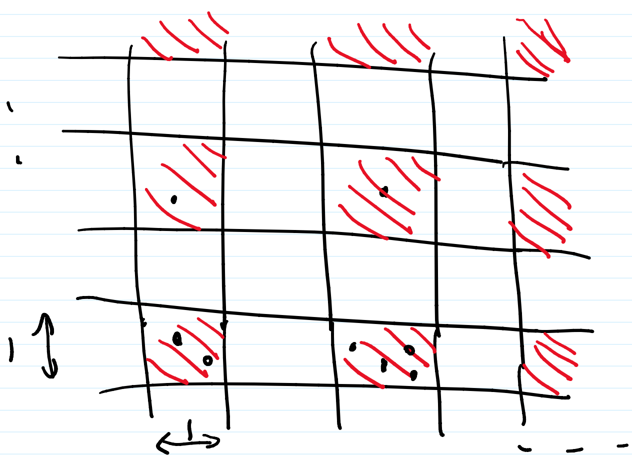
Equiv Problem: Given a set P of n pts in 2D, find a subset $Q \subseteq P$, maximizing $|Q|$,

$\forall q_1, q_2 \in Q$, $|q_1.x - q_2.x| > 1$
or $|q_1.y - q_2.y| > 1$.





Approx Algm 2: Grid



$$R_0 = \{ (x,y) : \lfloor x \rfloor \text{ is even } \& \lfloor y \rfloor \text{ is even} \}$$

Lemma The problem for $P \cap R_0$ can be solved in linear time.

Pf: Just pick one pt in each nonempty cell of R_0. □

let $R_1 = \{ (x,y) : \lfloor x \rfloor$ is even, $\lfloor y \rfloor$ odd $\}$

$R_2 = \{ (x,y) : \lfloor x \rfloor$ is odd, $\lfloor y \rfloor$ even $\}$

$R_3 = \{ (x,y) : \lfloor x \rfloor$ odd, $\lfloor y \rfloor$ odd $\}$

$O(n)$ time

1. for $i = 0$ to 3 do
2. $\quad Q_i =$ solution for $P \cap R_i$ (by lemma)
3. return the largest Q among $\{Q_0, \ldots, Q_3\}$.

Analysis: let $Q^* =$ opt sol'n.

Each pt belongs to one of the regions R_i.

$$|Q^*| = \sum_{i=0}^{3} \underbrace{|Q^* \cap R_i|}$$

$$\leq \sum_{i=0}^{3} |Q_i|$$

$$\leq 4\,|Q|$$

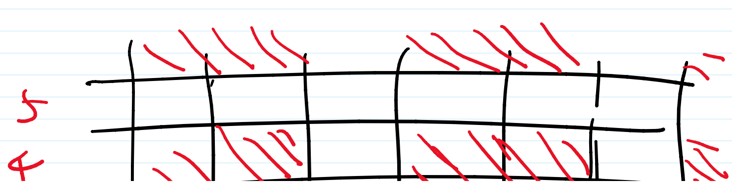
$$\Rightarrow \quad |Q| \geq \tfrac{1}{4} |Q^*| \qquad \text{approx factor } \tfrac{1}{4}.$$

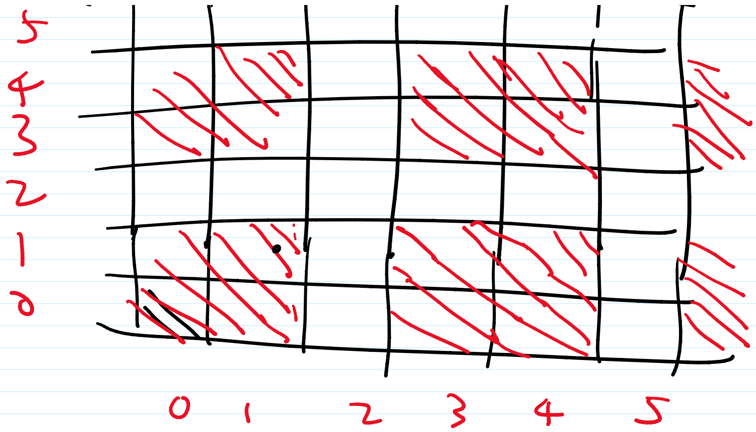
Hochbaum-Maass Alg/m (1985)

idea - again, shifted grid

fix const $k > 2$.

let $R_{ij} = \{ (x,y) : \lfloor x \rfloor \bmod k \neq i$ and $\lfloor y \rfloor \bmod k \neq j \}$





$k=3$

R_{22}

Lemma The problem for $P \cap R_{ij}$ can be solved in polytime.

Pf: can solve each $(k-1)\times(k-1)$ cell independently

in each $(k-1)\times(k-1)$ cell,

any feasible sol'n has size $\leq (k-1)^2$.

$\Rightarrow$ brute force $O\left(n^{(k-1)^2}\right)$ time. $\square$

polytime
1. for $i = 0, .., k-1$, $j = 0, .., k-1$
2. Q_{ij} = solution for $P \cap R_{ij}$ by lemma
3. return largest Q among these Q_{ij}'s.

Analysis: let $Q^* =$ opt sol'n.

Each pt belongs to $\underline{(k-1)^2}$ of the regions R_{ij}

$$(k-1)^2 |Q^*| = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} |Q^* \cap R_{ij}|$$

$$\leq \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} |Q_{ij}|$$

$$= k^2 |Q|$$

$\Rightarrow |Q| \geq \left(k-1 \right)^2 |Q^*|$

$\Rightarrow \quad |Q| \geq \left(\frac{k-1}{k}\right)^2 |Q^*|$

approx factor $\left(\frac{k-1}{k}\right)^2$

e.g. $k=2$: $\frac{1}{4}$

$k=3$: $\frac{4}{9}$

$k=4$: $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$k=5$: 0.64

$k=10$: 0.81

$k=100$: 0.9

$\vdots$

$\Rightarrow$ factor arb. close to 1.

polytime approx scheme (PTAS)