Ex1 SAT polytime-reduces to 3SAT.

Pf: rewrite $\alpha_1 \vee \dots \vee \alpha_k$ $(k > 3)$ $\qquad a \to b$, $\bar{a} \vee b$

as
$$\begin{aligned} &(y_1 \to \alpha_1 \vee \alpha_2) \\ \wedge &(y_2 \to y_1 \vee \alpha_3) \\ &\vdots \\ \wedge &(y_{k-3} \vee \alpha_{k-1} \vee \alpha_k) \end{aligned} \equiv \begin{aligned} &(\bar{y}_1 \vee \alpha_1 \vee \alpha_2) \\ \wedge &(\bar{y}_2 \vee y_1 \vee \alpha_3) \\ &\vdots \\ \wedge &(y_{k-3} \vee \alpha_{k-1} \vee \alpha_k). \end{aligned}$$

Ex2 0-1 Integer Linear Prog. (ILP)

3SAT polytime-reduces to 0-1 ILP.

Pf: rewrite $x_1 \vee \overline{x_2} \vee x_3$

as $x_1 + 1 - x_2 + x_3 \geq 1$.

$\square$

Ex3 Indep Set: given graph $G = (V, E)$, integer K, decide $\exists I \subseteq V$, $|I| \geq K$, $\forall u, v \in I \Rightarrow uv \notin E$.

$p \to q$
$\bar{q} \to \bar{p}$

(Cook '71)

3SAT polytime-reduces to Indep-Set.

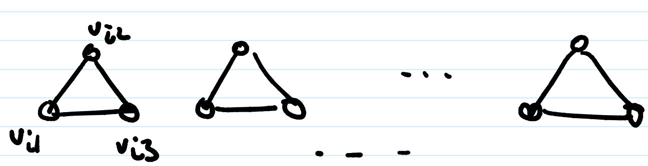
$u \in I \wedge v \in I \Rightarrow uv \notin E$

$uv \in E \Rightarrow u \in I^c$ or $v \in I^c$

Pf: Given 3CNF formula F with n vars, m clauses $C_1, \dots, C_m$
$C_i = \alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$

polytime

Construct graph $G = (V, E)$, K,



create m triangles $v_{i1} v_{i2} v_{i3}$, $i = 1..m$

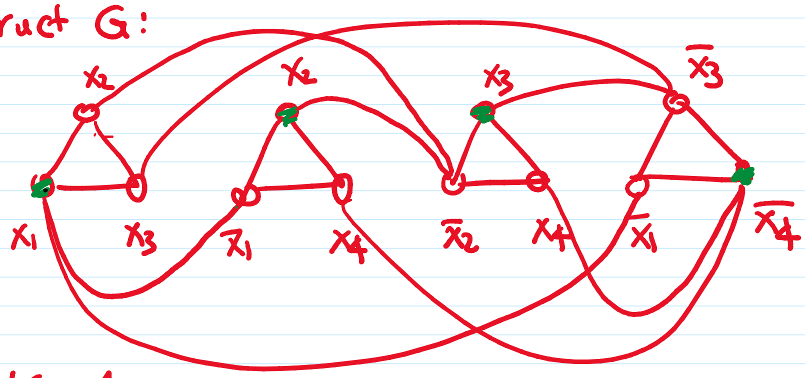
$\Rightarrow$ when $\alpha_{ij} = \overline{\alpha_{i'j'}}$, add edge $v_{ij} v_{i'j'}$.

$K = m$

$\vdots$

e.g. given $F = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$

Construct G:



$K=4$

Correctness Pf: $\exists$ assignment s.t. F evaluates to true

$\iff$ $\exists$ indep set in G of size $\geq K$

Pf: $(\Rightarrow)$ ...

$(\Leftarrow)$ ... $\square$

Ex 4 Vertex Cover:

Given graph $G=(V,E)$, integer K,

decide $\exists S \subseteq V$ s.t. $|S| \leq K$,

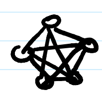
$\forall uv \in E, \Rightarrow u \in S$ or $v \in S$

(or both)

Indep Set polytime-reduces to Vertex-Cover

Pf: $K \to n-K$. $\square$

Ex 5 Clique:



Given graph G, k,

decide $\exists S \subseteq V$ s.t. $|S| \geq K$,

$\forall u,v \in S \Rightarrow uv \in E$.

Indep Set polytime-reduces to Clique

Pf: $G \nearrow G^c = (V, E^c)$, K. $\square$

Ex 6 Dir. Hamiltonian-Cycle:

Given dir. graph G,

Given dir. graph G,
decide $\exists$ cycle visiting every vertex once

(Karp '72)

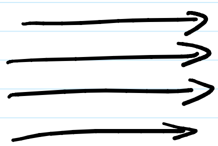
Vertex Cover polytime-reduces to Dir-Ham-Cycle.

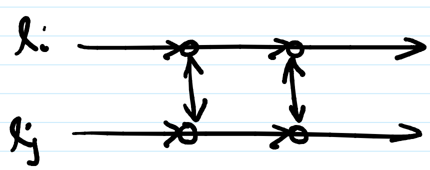
Pf: Given undir. graph $G=(V,E)$, K, $V=\{v_1,\dots,v_n\}$

Construct new dir. graph G':

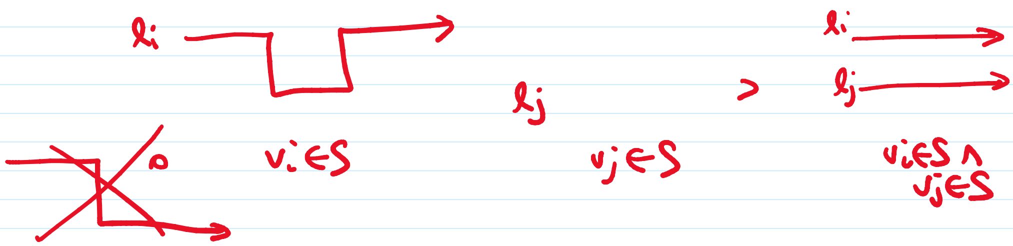
for each $v_i \in V$, create "lines" ℓ_i

for each $v_iv_j \in E$, add "gadget"



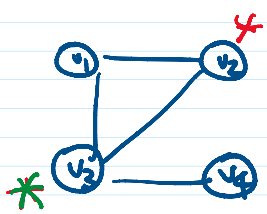


(only 3 ways to traverse gadget ...)

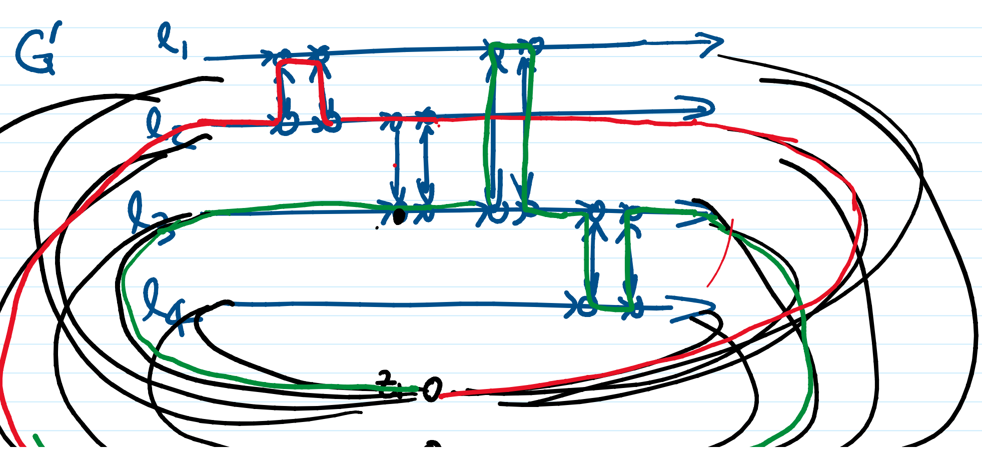


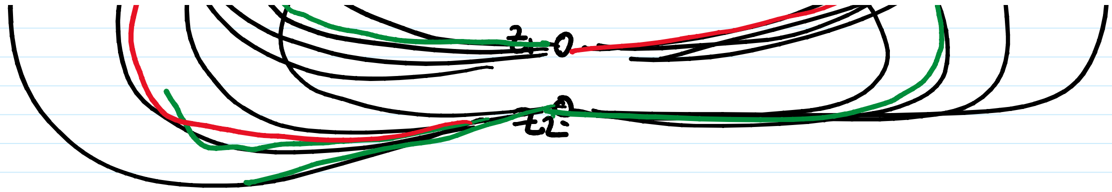
add K extra vertices $z_1,\dots,z_k$ connected to start & end of each line

e.g. G:



$K=2$





Correctness: $\exists$ vertex cover of size$^{(\leq)}$ k in G
$\Leftrightarrow$ $\exists$ Ham cycle in G'.

Pf: ($\Rightarrow$) ...
($\Leftarrow$) ... $\square$

Reductions can also prove polynomial conditional lower bds

"fine-grained complexity"

Ex diameter: Given unweighted undir graph $G=(V,E)$,
compute $D = \max_{u,v \in V} \delta_G(u,v)$.

shortest path length between u & v

Known: $O(mn)$ time by n BFSs
$O(n^{2.372})$ time

Thm (Roditty-Vassilevska Williams '13)
If diam could be solved in $O(mn^{0.99})$ time,
then k-SAT could be solved in $O(1.999^n)$ time for any const k

contradicting the "Strong Exponential Hypothesis" (SETH)