Ex

$$\begin{aligned} \max\ & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.}\ & 2x_1 + 3x_2 + x_3 \le 5 \\ & 4x_1 + x_2 + 2x_3 \le 11 \\ & 3x_1 + 4x_2 + 2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

rewrite

$$\begin{aligned} \max\ z &= 5x_1 + 4x_2 + 3x_3 \\ w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \end{aligned}$$

$$x_1, x_2, x_3, \underbrace{w_1, w_2, w_3}_{\text{called slack vars}} \ge 0$$

initial sol'n: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (0,0,0,5,11,8)$

$\bar{z} = 0$.

Iteration 1: increase $\bar{x}_1$ to 2.5

x_1 enters basis, w_1 leaves

Substitute $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$

$$\begin{aligned} \max\ z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \qquad 2.5/0.5 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \qquad 0.5/0.5 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \ge 0$$

new sol'n: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2.5, 0, 0, 0, 1, 0.5)$

$\bar{z} = 12.5$

Iteration 2: increase $\bar{x}_3$ to 1

x_3 enters basis, w_3 leaves

substitute $x_3 = 1 + 3w_1 + x_2 - 2w_3$

$$\begin{aligned} \max\ z &= 13 - w_1 - 3x_2 - w_3 \qquad \le 13 \\ x_1 &= 2 - 2w_1 - 2x_2 + w_3 \end{aligned}$$

$$\max z = 13 - w_1 - 3x_2 - w_3 \quad \leq 13$$

$$x_1 = 2 - 2w_1 - 2x_2 + w_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$x_3 = 1 + 3w_1 + x_2 - 2w_3$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

new sol'n: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2, 0, 1, 0, 1, 0)$

$\bar{z} = 13$

done!

correctness obvious

Analysis: each iteration takes $O(mn)$ time

\# iterations?

trivial upper bd $\leq$ total # vertices/bases

because each vertex is visited at most once } if no degeneracies

$\leq \binom{m+n}{n}$

$\leq 2^{m+n}$ exponential!

(if there are degeneracies, could loop forever ...)

Can be avoided e.g. by using "Bland's rule") choose pivot var: lowest index in case of ties)

Klee-Minty '73: $\exists$ examples which require exponential time for simplex algm for various pivot-choosing rules.

BUT in practice, # iterations tend to be small

How to find initial basic feasible sol'n ?

(if origin is not feasible)

add an extra var x_0

e.g. max $x_1 + x_2$ $\Rightarrow$ max $-x_0$ aux LP

s.t. $2x_1 - x_2 \leq 2$ $\qquad$ $-x_0 - 2x_1 - x_2 \leq 2$

$x_1 - 5x_2 \leq -4$ $\qquad$ $-x_0 + x_1 - 5x_2 \leq -4$

$x_1, x_2 \geq 0.$ $\qquad$ $x_0, x_1, x_2 \geq 0$

init sol'n: $(\bar{x}_0, \bar{x}_1, \bar{x}_2) = (4, 0, 0)$

If opt sol'n to aux LP has $x_0 \neq 0$, LP is infeasible (no sol'n)

Duality

how to certify optimality?

$$\max\ 5x_1 + 4x_2 + 3x_3 = z$$
$$\text{s.t.}\quad 2x_1 + 3x_2 + x_3 \leq 5 \quad \cancel{0.6} \quad 1 \quad y_1$$
$$4x_1 + x_2 + 2x_3 \leq 11 \quad \cancel{0.5} \quad 0 \quad y_2$$
$$3x_1 + 4x_2 + 2x_3 \leq 8 \quad \cancel{0.7} \quad 1 \quad y_3$$
$$x_1, x_2, x_3 \geq 0$$

8.5
5.6

1.2
+2
+2.1

$\Rightarrow z \leq 5.3x_1 + 4.1x_2 + 3x_2 \leq 5 \cdot 0.6 + 11 \cdot 0.5 + 8 \cdot 0.7 = 14.1.$

$\Rightarrow z \leq 5x_1 + 7x_2 + 3x_3 \leq 13$

how to find the best linear combination?
it's another LP !!!

e.g. let y_1, y_2, y_3 be multipliers

min $5y_1 + 11y_2 + 8y_3$

$$\left.\begin{aligned} \text{s.t. } & 2y_1 + 4y_2 + 3y_3 \geq 5 \\ & 3y_1 + y_2 + 4y_3 \geq 4 \\ & y_1 + 2y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}\right\} \text{"dual LP"}$$

Def Given LP, in standard form

$$\begin{aligned} \max\ & c_1x_1 + \dots + c_nx_n \\ \text{s.t. } & a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\ & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m \\ & x_1, \dots, x_n \geq 0 \end{aligned} \quad \xRightarrow{\text{dual LP}} \quad \begin{aligned} \min\ & b_1y_1 + \dots + b_my_m \\ \text{s.t. } & a_{11}y_1 + \dots + a_{m1}y_m \geq c_1 \\ & \vdots \\ & a_{1n}y_1 + \dots + a_{mn}y_m \geq c_n \\ & y_1, \dots, y_m \geq 0. \end{aligned}$$

primal

Obs value of any feasible sol'n to primal LP $\leq$ value of any feasible sol'n to dual LP.

Pf: Straightforward. $\square$

Thm (LP Duality)

max value of primal LP = min value of dual LP.

Pf: ($\leq$) from Obs.

($\geq$) run Simplex alg'm.

get multipliers from the final slack form

$$\max\ z = \alpha - \beta_1 w_1 - \dots - \beta_m w_m - \gamma_1 x_1 - \dots - \gamma_n x_n$$

multipliers we want. $\square$