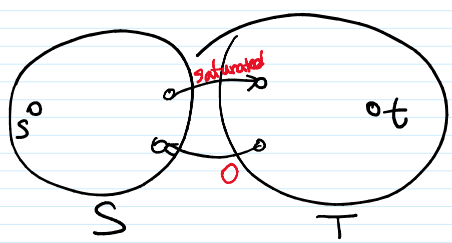
Thm 2 (max-flow min-cut thm) value of max flow = capacity of min (s,t)-cut.

this Friday Apr 4: Review Session
W 109 TH (Turner Hall)

Pf of Thm 1 & 2:

Let f be flow with no aug path.

i.e. no path from s to t in residual graph G_f.



Let S be all vertices reachable from s.

$T = V - S$.

$\forall u \in S,\ v \in T,\quad (u,v) \in E \Rightarrow f(u,v) = c(u,v)$

$\qquad\qquad\qquad\quad (v,u) \in E \Rightarrow f(u,v) = 0$

$$\text{max flow value} \geq |f| = \sum_{u \in S, v \in T} f(u,v) = \sum_{\substack{u \in S, v \in T \\ (u,v) \in E}} c(u,v) - \cancel{\sum_{\substack{u \in S, v \in T \\ (v,u) \in E}} 0} = \text{capacity of } (S,T) \geq \text{capacity of min cut} \geq \text{value of max flow}$$

$\square$

Runtime Analysis:

each iteration takes $O(m)$ time by BFS/DFS

if capacities are integers in $\{1, .., U\}$,

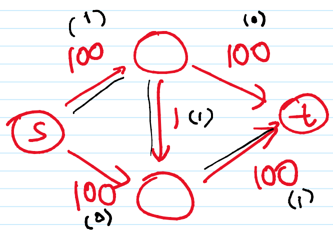
$\# \text{ iterations} \leq |f^*| \leq nU$

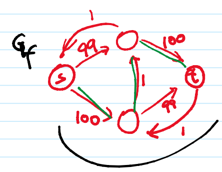
($|f^*|$: max flow value)

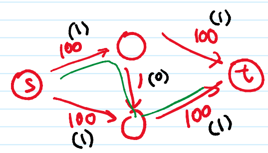
$\Rightarrow O(mnU)$ total time

$\Rightarrow \quad O(mnU)$ total time
(pseudo-polynomial)
not strongly polynomial

Bad Ex:







(if not integers, may not terminate!)

Corollary (Integrality Thm)
If all capacities are integers,
opt flow has integer values.

Edmonds-Karp Version I ('72)

always pick aug path with largest bottleneck value
by variant of Dijkstra
e.g. $O(m \log n)$

Analysis:
Claim # iterations $\leq O(m \log |f^*|)$

Pf:
Fact (Flow Decomposition Thm)
any flow can be expressed
as a sum of $\leq m$ cycle flows
& $\leq m$ path flows

let f_i^* be max flow in G_f in i^{th} iteration
$\searrow$ decompose into ~~$\leq m$ cycle flows~~ remove
& $\leq m$ path flows in G_f
$\Rightarrow \quad \exists$ path flow with $\geq \frac{|f_i^*|}{m}$ value
aug path in G

$$|f_{i+1}^*| \leq |f_i^*| - \frac{|f_i^*|}{m} = \frac{m-1}{m} |f_i^*|$$

$$\Rightarrow \# \text{ iterations} \leq \log_{\frac{m}{m-1}} |f^*|$$

$\log |f^*|$

$\Rightarrow$ # iterations $\leq \log_{\frac{m}{m-1}} |f^*|$

$= \dfrac{\log |f^*|}{\log\left(1+\frac{1}{m-1}\right)}$

$= O(m \log |f^*|)$. □

$\Rightarrow O(m \log n \cdot m \log |f^*|)$

$\leq \boxed{O(m^2 \log^2 (nU))}$.

Edmonds-Karp Version 2:

always pick shortest aug path
edges

each iter. $O(m)$ time by BFS

Claim # iterations $\leq mn$

Pf omitted. □

$\Rightarrow$ total time $\boxed{O(m^2 n)}$

Later History:

Dinic '70 $O(mn^2)$

Goldberg-Tarjan '86 $O(mn \log n)$

Goldberg-Rao '98 $O(m^{3/2} \log^2(nU))$
$O(mn^{2/3} \log^2(nU))$

Madry '16 $\tilde{O}(m^{10/7} U^{1/7})$

⋮

'21 $\tilde{O}(m^{\frac{3}{2}-\frac{1}{328}} \log U)$

'22 $\tilde{O}(m^{\frac{3}{2}-\frac{1}{58}} \log U)$

Chen, Kyng, Liu, Peng, Gütenberg, Sachdeva '22 (112 pp) $O(m^{1+\varepsilon} \log U)$

Generalization: Min-Cost Flow

find flow f with $|f| = d$
minimizing $\sum_{(u,v)\in E} \text{cost}(u,v) \cdot f(u,v)$