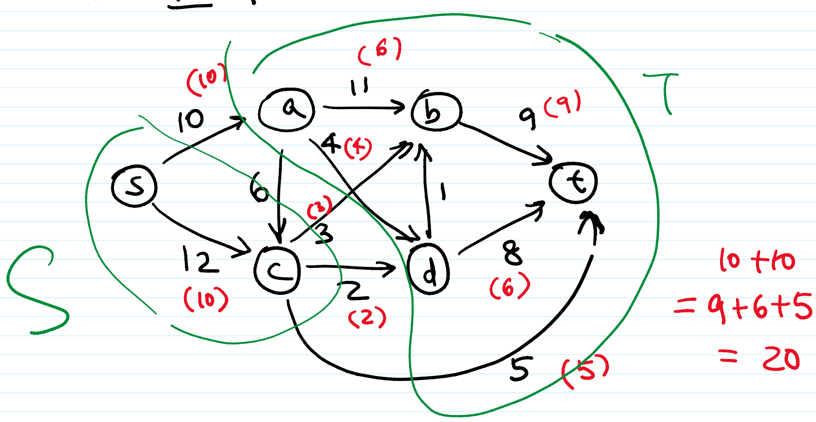
March 28, 2025 12:18 PM

## Problem max flow

"source" (in-deg 0) "sink" (out-deg 0)

Given dir graph $G=(V,E)$, $s,t \in V$,
where each $e \in E$ has a capacity $c(e) > 0$,

find flow from s to t with max value




$10+10$
$= 9+6+5$
$= 20$

**Def** A flow is $f: E \to \mathbb{R}$ s.t.

$$\forall (u,v) \in E, \quad 0 \le f(u,v) \le c(u,v)$$

flow conservation
$$\forall v \in V - \{s,t\}, \quad \sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$$

$$\text{Value of } f = |f| = \sum_{v:(s,v)\in E} f(s,v)$$

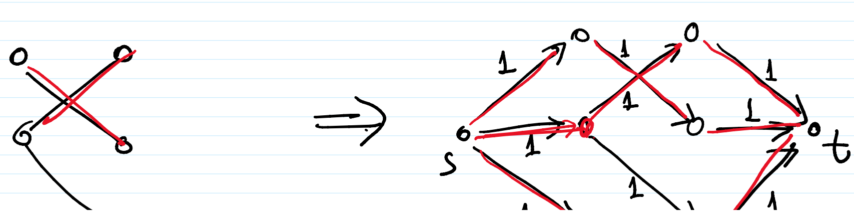
$$\left( = \sum_{v:(v,t)\in E} f(v,t) \right)$$

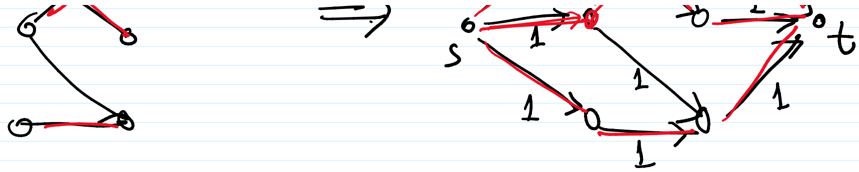
**Convention:** make $f(u,v) = -f(v,u)$ if $(v,u) \in E$

& $f(u,v) = 0$ if $(u,v), (v,u) \notin E$.

( then flow conservation means $\sum_{w \in V} f(v,w) = 0$. )

**Appl 1:** max bipartite matching

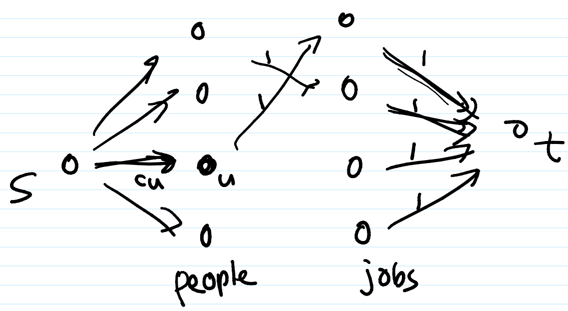





by integrality thm,
opt flow is integral

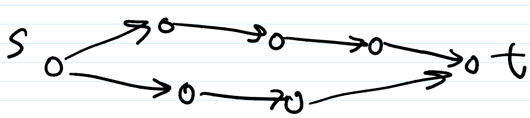
**Appl 2:** assignment problems

assign jobs to people
each person $u$ can do $\leq c_u$ jobs
max # assignments

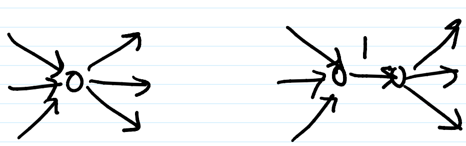



**Appl 3** - given unweighted dir graph, s, t,

find max # of edge-disjoint paths
from s to t




(for vertex-disjoint,

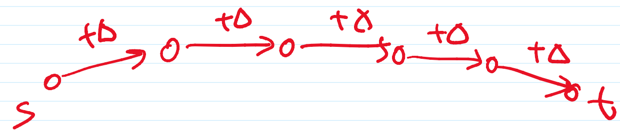


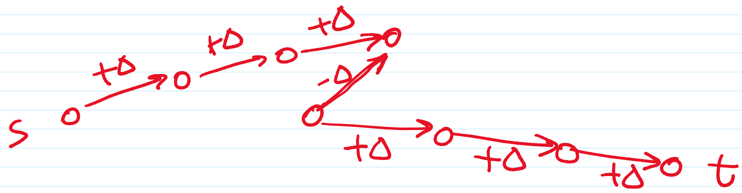

)

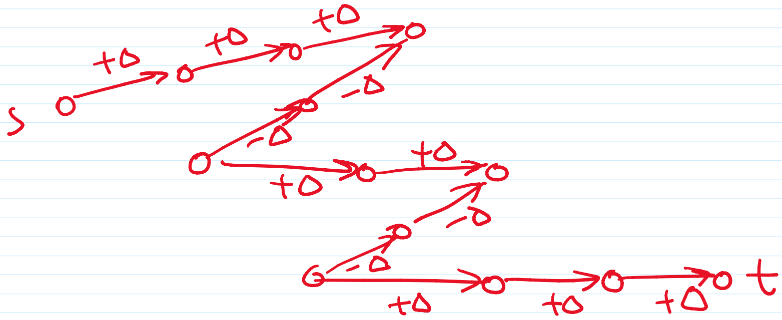
Etc.

## Ford-Fulkerson Alg'm ('56)

idea - local improvement again!







**Def** Let $f$ be a flow.

Define residual graph $G_f$ :

if $(u,v) \in E$, $f(u,v) < c(u,v)$, (not saturated)

add $(u,v)$ in $G_f$ with capacity $c_f(u,v) = c(u,v) - f(u,v)$

if $(v,u) \in E$, $f(v,u) > 0$,

add $(u,v)$ in $G_f$ with capacity $c_f(u,v) = f(v,u)$

**Alg'm:**

$f$ = zero flow

repeat {

find an s-to-t path $p$ in $G_f$ ← called augmenting path

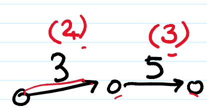
if not exist, return $f$

$\Delta$ = min edge capacity in $p$ in $G_f$ ← bottleneck value

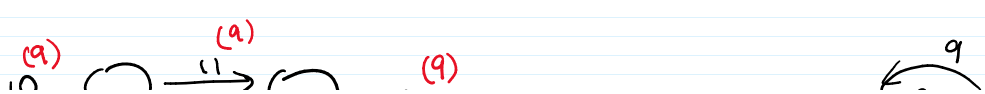
$\forall (u,v) \in p$, $f(u,v) \mathrel{+}= \Delta$

$f(v,u) \mathrel{-}= \Delta$

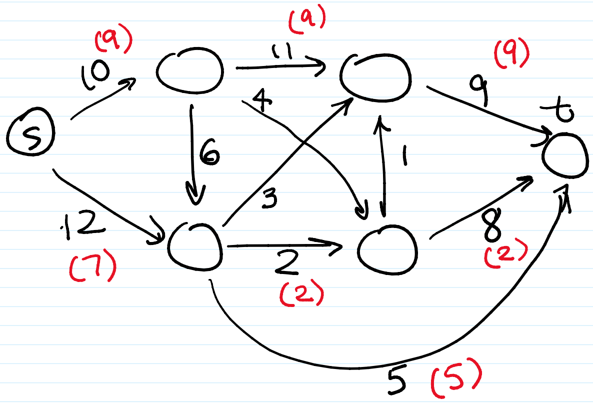
}



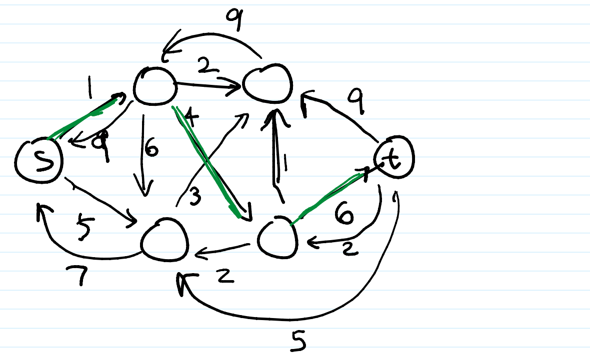
**Ex**



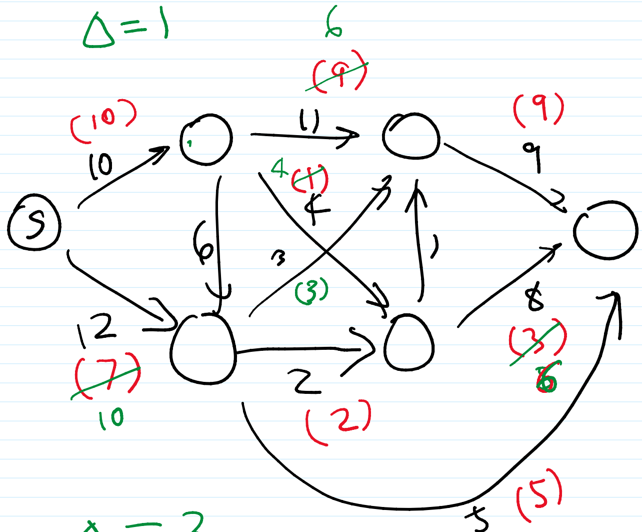
Ex



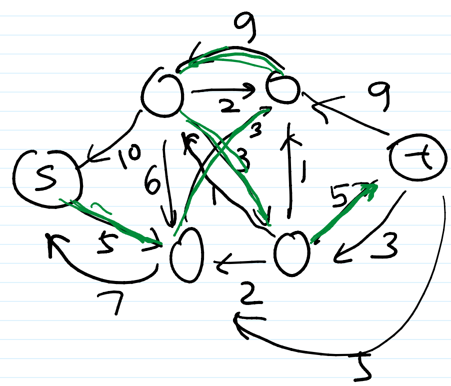
$G_f$



$\Delta = 1$



$G_f$



$\Delta = 3$

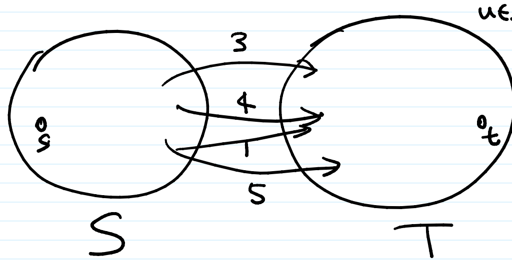
## Correctness:

**Thm 1** flow $f$ is optimal $\iff$ it has no augmenting path.

How to certify optimality?

**Def** An s-t cut is a partition $(S,T)$ of $V$ with $s \in S$, $t \in T$

The capacity of $(S,T) = \sum_{u \in S, v \in T} c(u,v)$



**Obs** value of any flow $\leq$ capacity of any $(s,t)$-cut.

Pf: $|f| = \sum_{u \in S, v \in T} f(u,v) \leq \sum_{\substack{u \in S, v \in T \\ (u,v) \in E}} c(u,v) - \cancel{\sum_{\substack{u \in S, v \in T \\ (v,u) \in E}} 0}$ $\square$

**Thm 2** value of max flow $=$ capacity of min $(s,t)$-cut.

Pf of Both Thms:

$\vdots$