$$= \sum_{x \in S - \{y\}} \underline{\Pr[h_{a,b}(x) = h_{a,b}(y)]}$$

$$\le O\left(n \cdot \frac{1}{m}\right)$$

Set $m \approx n$ $\Rightarrow$ $\boxed{O(1)}$ expected query time
$\boxed{O(n)}$ space
$O(n)$ preproc time

Q: can we get worst-case query time?

yes (in static case)

Method by Fredman, Komlós, Szemerédi '84

Part I. Perfect hashing

$$E_{a,b}\left[\#\text{ "colliding" pairs } (x,y) \in S \times S \ (x \neq y) \text{ s.t. } h_{a,b}(x) = h_{a,b}(y)\right]$$

$$= E_{a,b}\left[\sum_{\substack{(x,y)\in S\times S \\ x \neq y}} I_{x,y}\right] \qquad I_{x,y} = \begin{cases} 1 & \text{if } h_{a,b}(x) = h_{a,b}(y) \\ 0 & \text{else} \end{cases}$$

linearity of exp.

$$\sum_{\substack{(x,y)\in S\times S \\ x \neq y}} E_{a,b}[I_{x,y}]$$

$$= \sum_{\substack{(x,y)\in S\times S \\ x \neq y}} \Pr_{a,b}[h_{a,b}(x) = h_{a,b}(y)]$$

$$\le \boxed{O\left(n^2 \cdot \frac{1}{m}\right)}$$

Set $m = cn^2$ $\quad = O\left(\frac{1}{c}\right)$

Markov's ineq: $X \ge 0$, $\mu = E(X)$

Set $m = cn^2$ $\qquad = O\left(\frac{1}{c}\right)$

Markov's ineq: $\mu = E(X)$

$\Pr[X \geq a\mu] \leq \frac{1}{a}$

$\Pr[X \geq t] \leq \frac{\mu}{t}$

$\Pr\{\#\text{collision pairs} \geq 1\}$

$$\leq \frac{O(1/c)}{1} = O\left(\frac{1}{c}\right)$$

$$\leq 0.1$$ by setting c large const.

repeat until success (no collision)

$\Rightarrow$ expected # trials $\frac{1}{0.9} = O(1)$

$\Rightarrow$ $O(n^2)$ expected prep time

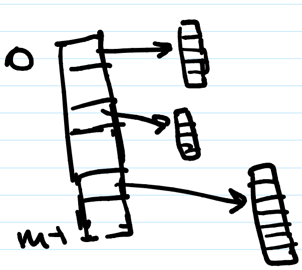
$O(n^2)$ worst-case space

$O(1)$ worst-case query time

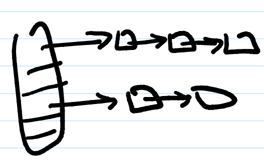
($\Rightarrow$ this prove existence of perfect hashing)

Part II. reduce space by bootstrapping!

Store each bucket $A[i]$ by a perfect hash table (Part I)



"2-level hash table"



Space $O\left(m + \sum_{i}^{m-1} |A[i]|^2\right)$

space $O\left(m + \sum_{i=0}^{\cdots} |A[i]|^2\right)$

$= O(m + \#\text{ colliding pairs})$

$\left(\#\text{ colliding pairs} = \sum_i \binom{|A[i]|}{2} \approx \sum |A[i]|^2\right)$

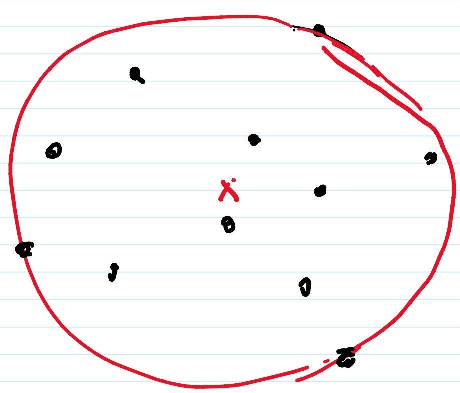
$E[\text{space}] = O\left(m + \frac{n^2}{m}\right)$

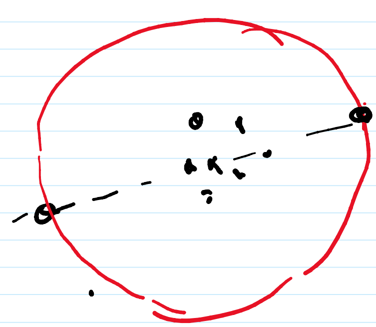
Set $m = n$ $\quad = \boxed{O(n)}$ can be made worstcase by repeating

worst-case query time $\boxed{O(1)}$

expected preproc time $\boxed{O(n)}$

Problem Given set S of n points in 2D, compute the min circle enclosing S.





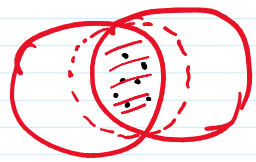
Obs C^* is unique & has 3 pts on boundary

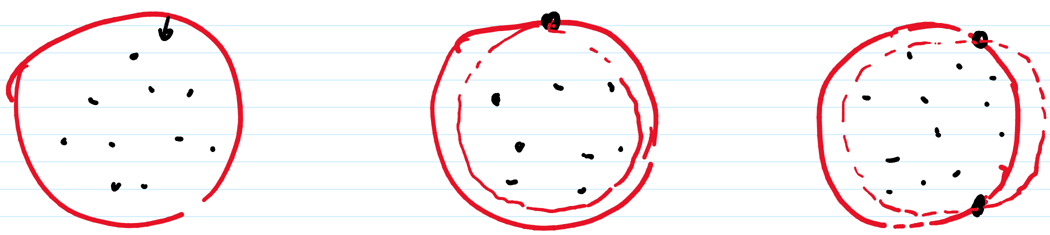
or 2 pts on boundary as "diametrical pair"

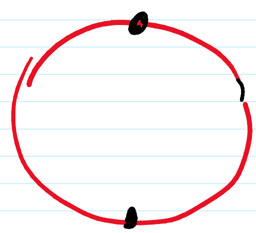
or 2 pts on boundary
as "diametrical pair"

$\Rightarrow$ brute force algo'm $O(n^3 \cdot n) = O(n^4)$ time

Pf Sketch: By contradiction.

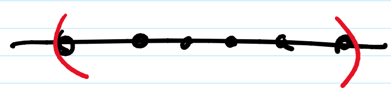






$\square$

Digression:



Subproblem (1D) Given n numbers $S = \{x_1, \ldots, x_n\}$,
find the min.

Standard "Incremental" Alg'm:

0. randomly permute $x_1, \ldots, x_n$
1. ans $= \infty$
2. for $i = 1$ to n
3. if $x_i <$ ans then
4. ans $= x_i$. $\leftarrow$ (*)

Q: how many switches (*)?

Q: how many ^expected # switches (*)?

rewrite alg'm:

min(S): // "backwards" version

0. if $S = \emptyset$ return ∞
1. pick $x \in S$ randomly
2. ans = min($S - \{x\}$)
3. if $x <$ ans then
4. ans = x ← (*)
5. return ans

$$\Pr[(*) \text{ is done}] = \Pr[x = \min(S)] = \frac{1}{n}.$$

$\Rightarrow$ $T(n) = T(n-1) + \frac{1}{n}$ by linearity of expectation

$$\Rightarrow T(n) = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{2} + 1$$

$= n^{th}$ Harmonic number

$= \ln n + O(1)$

$= O(\log n)$.