Approximation Algorithms — III (Randomized Rounding)

SATisfiability.

Given:- 1) Set of variables $x_1, x_2, \ldots, x_n$
2) Set of clauses $C_1, C_2, \ldots, C_m$.
$C_i = (x_i \vee \neg x_{i+1} \vee \cdots)$

Goal:→ Find an assignment that maximizes # clauses satisfied.

Randomized Alg:

$$x_i = \begin{cases} T & \text{w.p. } 1/2 \\ F & \text{w.p. } 1/2 \end{cases}$$

$$\Pr[C_r \text{ is satisfied}] = 1 - \left(\tfrac{1}{2}\right)^{l_r} = 1 - 2^{-l_r}$$

C_r contains l_r literals.

$$E[\#\text{clauses satisfied}] = \sum_{C_r} (1 - 2^{-l_r})$$

if every clause contains atleast 3 literals if every clause contains atleast 3 literals

then in expectation $\frac{7}{8}$ clauses are satisfied.

SATisfiability ILP

$z_j = 1$ if C_j is satisfied
$\quad\ \ 0$ otherwise

$y_i = 1$ if $x_i = T$
$\quad\ \ 0$ if $x_i = F$

meaning of z + y

ILP

LP

$$\max \sum_j z_j$$

P_j = variables that occur in C_j without neg

$$\sum_{i \in P_j} y_i + \sum_{r \in N_j} (1 - y_i) \geq z_j$$

N_j = variables that occur in C_j with neg

$$0 \leq z_j \leq 1.$$

$$y_i \in \{0, 1\}$$

$$0 \leq y_i \leq 1. \quad \forall i \in [n]$$

ALG

ALG

1) Solve SAT-LP. Let z^*, y^* be the OPT soln to the LP

2) Rand $z^* \& y^*$

↳ Set $x_i = T$ with probability y_i^*

$= F$ with probability $1-y_i^*$

$\Pr[\text{Clause } C_j \text{ is unsatisfied}]$ $\quad l_j \quad P_j, N_j$

$$= \left(\left(\prod_{i \in P_j} (1-y_i^*) \cdot \prod_{i \in N_j} y_i^*\right)^{\frac{1}{l_j}}\right)^{l_j}$$

GM ≤ AM

Fact

$$\left(\prod_i x_i\right)^{1/n} \le \frac{1}{n}\sum_i x_i$$

$$\le \left\{\frac{1}{l_j}\left\{\sum_{i \in P_j}(1-y_i^*) + \sum_{i \in N_j} y_i^*\right\}\right\}^{l_j}$$

$$= \left\{\frac{1}{l_j}\left\{|P_j| - \sum_{i \in P_j} y_i^* + \sum_{i \in N_j} y_i^*\right\}\right\}^{l_j}$$

$$= \left\{\frac{1}{l_j}\left\{|P_j| - \sum_{i \in P_j} y_i^* + |N_j| - \sum_{i \in N_j}(1-y_i^*)\right\}\right\}^{l_j}$$

$$= \left\{ \frac{1}{l_j} \left\{ l_j - \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right\} \right\}^{l_j}$$

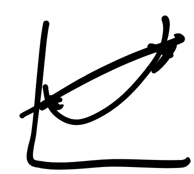
$$\leq \left\{ \frac{1}{l_j} \left(l_j - z_j^* \right) \right\}^{l_j}$$

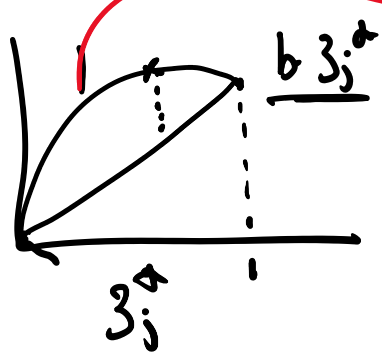
$$= \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

$$\Pr[C_j \text{ is satisfied}] = \left(1 - \left(1 - \frac{z_j^*}{l_j} \right)^{l_j} \right)$$

$E[\#\text{clauses satisfied}]$

$$= \sum_j \underbrace{\left(1 - \underbrace{\left(1 - \frac{z_j^*}{l_j} \right)^{l_j}}_{\text{convex fn}} \right)}_{\text{concave fn}}$$





$$b = \left(1 - \left(1 - \frac{1}{l_j} \right)^{l_j} \right)$$

$E[\#\text{clauses satisfied}]$

$$\geq \sum_j b \cdot z_j^{\alpha} \qquad b = \max_j \left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)$$

$$= b \underline{\sum_j z_j^{\alpha}}$$

$$\geq \underline{b \text{ OPT}} \qquad \underline{\left\{1 - \left(1-\frac{1}{l_j}\right)^{l_j}\right\}}$$

$l_j = 1 \qquad b = 1$

$l_j = 2 \qquad b = 3/4$

$\vdots$

$l_j \to \infty \qquad b = 1 - 1/e \approx 0.63 \qquad \left(1-\frac{1}{n}\right)^n \underset{n\to\infty}{\leadsto} \frac{1}{e}$

A_1 = solution corresponding to unbiased rounding

A_2 = solution - - - - - biased rounding

$$E\left[\max\left(\#\text{clauses satisfied by } A_1, \#\text{clauses by } A_2 \ldots\right)\right]$$

$$\geq E\left[\frac{1}{2} \cdot \#\text{clauses satisfied by } A_1 + \frac{1}{2} \cdot \#\text{clauses satisfied by } A_2\right]$$

$$\geq E\left[\frac{1}{2}\cdot \#\text{clauses satisfied by } A_1 + \frac{1}{2}\cdot \#\text{clauses satisfied by } A_2\right]$$

$$= \frac{1}{2}\cdot\left\{ \underline{E\left[\#\text{clauses}_{A_1}\ldots\right]} + E\left[\#\text{clauses satisfied by } A_2\right]\right\}$$

$$\geq \frac{1}{2}\left\{ \sum_j z_j^*\underline{(1-2^{-l_j})} + \sum z_j^*\cdot \underline{\left(1-\left(1-\tfrac{1}{l_j}\right)^{l_j}\right)}\right\}$$

$$= \sum_j z_j^* \cdot \left\{ \underbrace{\frac{(1-2^{-l_j}) + \left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)}{2}}_{f(l_j)} \right\}$$

$$f(1) = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$$

$$f(2) = \frac{\frac{3}{4}+\frac{3}{4}}{2} = \frac{3}{4}$$

$$f(\geq 3) = \frac{\geq \frac{7}{8} + \geq(1-\frac{1}{e})}{2} \geq \frac{3}{4}$$

$$E[\#\text{clauses satisfied}] \geq \frac{3}{4}\cdot\sum_j z_j^*$$

$\geq \frac{3}{4} \cdot OPT$

SET-COVER

Given :- 1) Set $U = e_1, e_2 \ldots, e_n$ elements
2) Sets $S_1, S_2 \ldots S_m$ $S_i \subseteq U$
3) weights w_i for sets S_i

Goal :- Choose $T = \{S_{i_1}, S_{i_2} \ldots, S_{i_{|T|}}\}$

s.t. $\bigcup_{S \in T} S = U$

& minimize $\sum_{S \in T} w_S$.

SET-COVER -ILP

$x_S = \begin{cases} 1 & \text{if set } S \in T \\ 0 & \text{otherwise} \end{cases}$

SET-COVER LP

$$\sum_S x_S \cdot w_S$$

$$\sum_{S:\, e \in S} x_S \geq 1 \quad \forall\, e \in U$$

$$x_S \in \{0, 1\}$$

$$0 \leq x_S \leq 1$$

ALG

ALG

$0 \le x_S \le 1$

1) Solve SET-COVER LP. Let x^* be the optimum sol^n

Repeatedly for t iterations

2) Round x^*.

Select set S w.p. x_S^*

3) Return set as union of all iteration

$$E[\text{cost of the sol}^n] = \sum_S w_S \cdot x_S^*$$

Pick an element e.

$$\Pr[e \text{ is not covered}] \qquad |\{S \mid e \in S\}| = k$$

$$= \left(\left(\prod_{S: e \in S} (1 - x_S^*) \right)^{\frac{1}{k}} \right)^k$$

$$\le \left(\frac{1}{k} \sum_{S: e \in S} (1 - x_S^*) \right)^k \qquad (GM \le AM)$$

$$= \left(1 - \sum_{S: e \in S} \frac{x_S^*}{k} \right)^k$$

$$\le \left(1 - \frac{1}{k}\right)^k \le \left(\frac{1}{e}\right)^t = \frac{1}{e^t}$$

$\Pr[\exists\, e \in U$ that is not covered$]$

$$\leq \sum_{e \in U} \Pr[e \text{ is not covered}] \quad // \text{union bound}$$

$$\leq \frac{n}{e^t}$$

$$\Pr[S \text{ is a set cover}] \geq 1 - \frac{n}{e^t} \qquad 1 - 1/n^2$$

set $t = 3 \lg n$

Approximation factor $= t \cdot OPT$

Algorithm should be succesful with probability $1-\varepsilon \Rightarrow$

$$\frac{n}{e^t} \leq \varepsilon \Rightarrow t \in O(\lg \tfrac{n}{\varepsilon})$$

Approximation factor $= O(\lg \frac{n}{\varepsilon})$

More on RR 4 TSP?