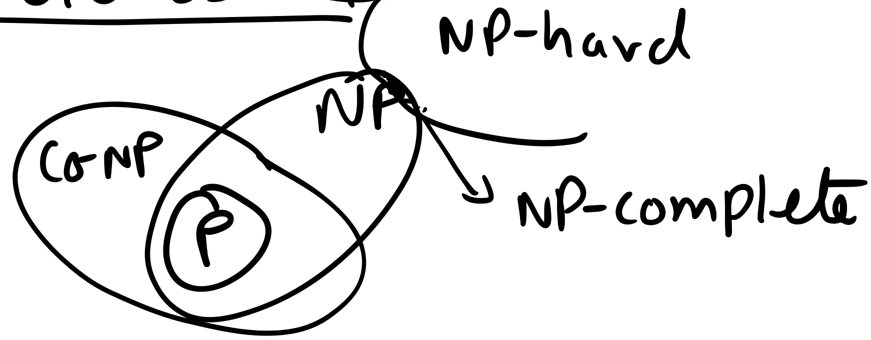
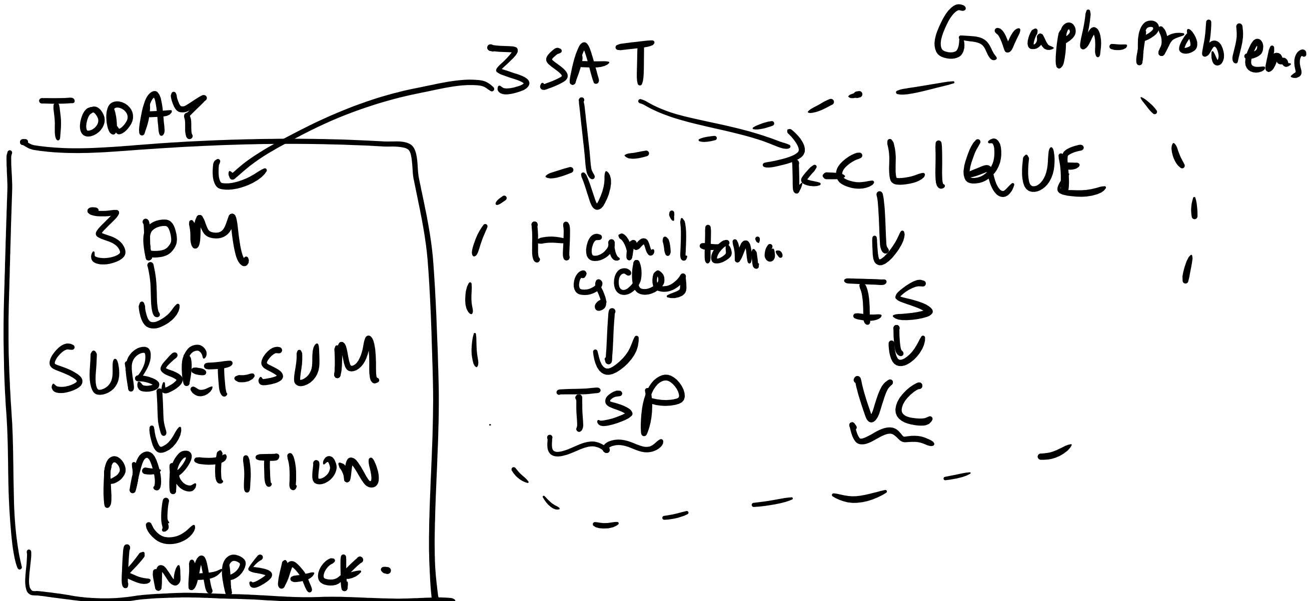
Thursday, April 13, 2023 2:01 PM

NP-Completeness -II



$$\text{CNF-SAT} \le_p \text{3SAT}$$



3DM Given:- 1) Three disjoint sets of vertices X, Y, Z

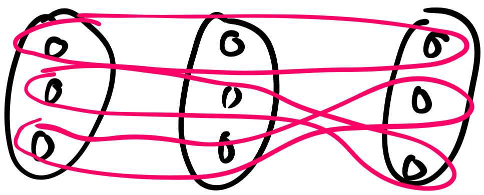
2) $|X| = |Y| = |Z| = n$

3) Set of Triples T.
each $t \in T = (x_i, y_j, z_k)$ with $x_i \in X$, $y_j \in Y$, $z_k \in Z$

Question:- $\exists\ t_1, t_2, \ldots, t_n$

s.t $\quad t_i \cap t_j = \phi \quad \forall i,j \in (n)$

$$\bigcup_i t_i = X \cup Y \cup Z$$



Thm:- 3DM is NP-complete

1) 3DM $\in$ NP.
2) 3SAT $\leq_p$ 3DM.

$\hookrightarrow$ $x_1, x_2, \cdots, x_n$ } variables
$c_1, c_2, \cdots, c_m$ } clauses

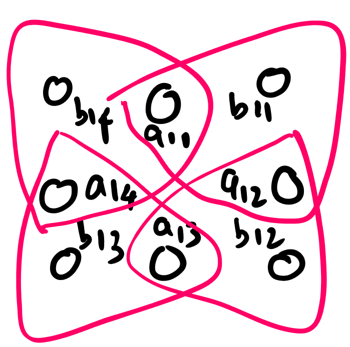
Variable-Gadgets.

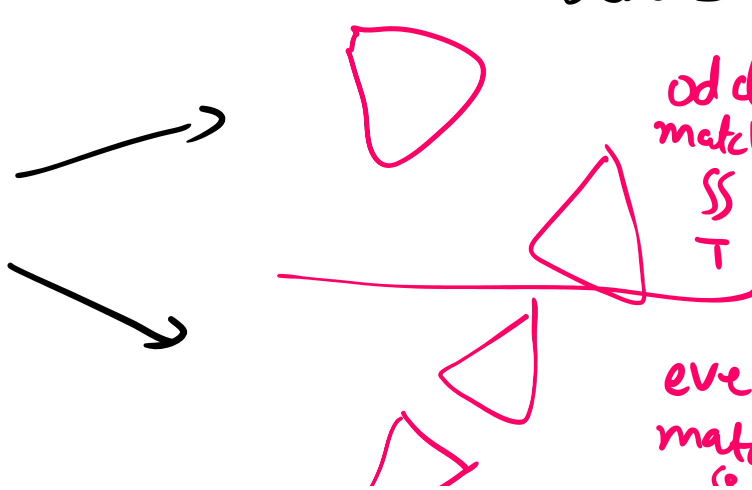
x_i →
- core-vertices $= a_{i1}, a_{i2}, \cdots, a_{i2m}$.
- tip-vertices $= b_{i1}, b_{i2}, \cdots, b_{i2m}$
- triples $= \bigcup_j (a_{ij}, a_{ij+1}, b_{ij})$

indices are modulo $2m$

$m = 2$

x_1

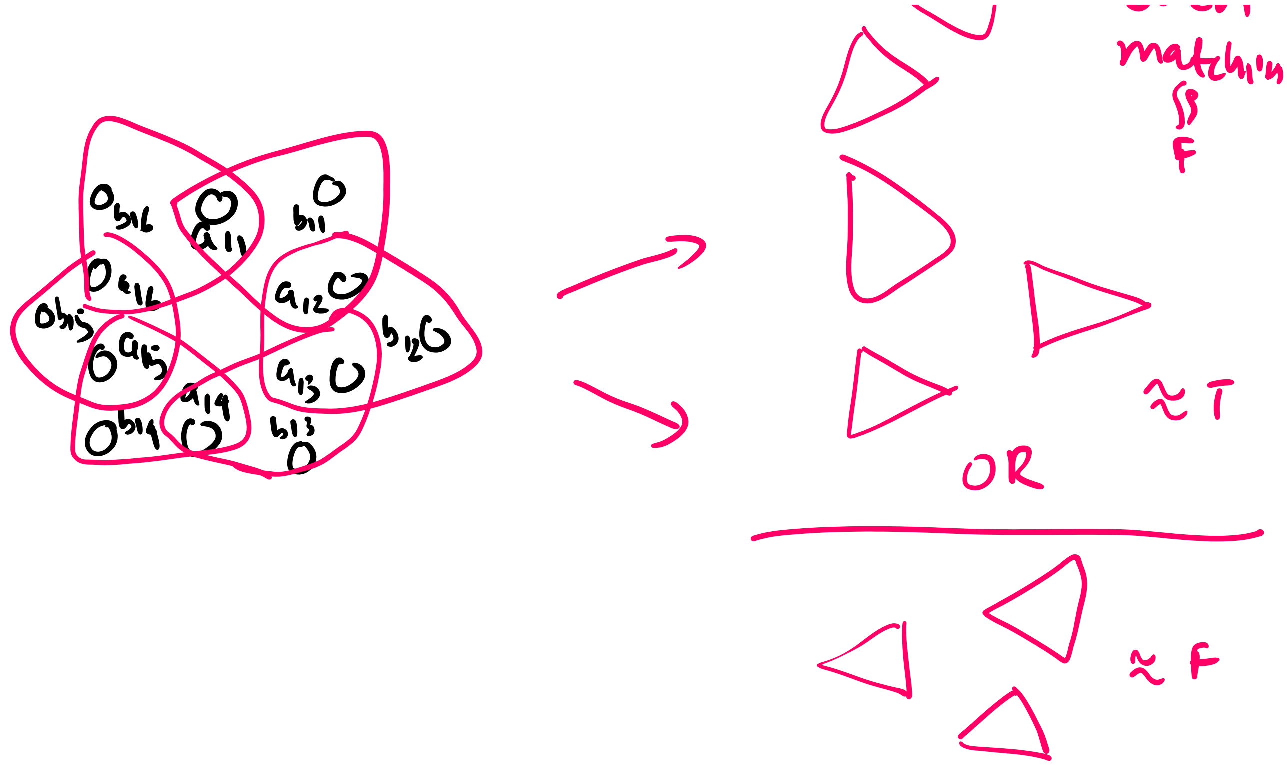




odd matching
$\Updownarrow$
T

even matching

$m = 3$



Clause-gadgets

For each clause $C_r = (l_i \vee l_j \vee l_k)$

1) 2 vertices C_r, C'_r

2) Add $(C_r, C'_r, \underline{b_{i,2r-1}})$ if $l_i = x_i$

$(C_r, C'_r, b_{i,2r})$ if $l_i = \neg x_i$

3) - — .

4) - — - ——

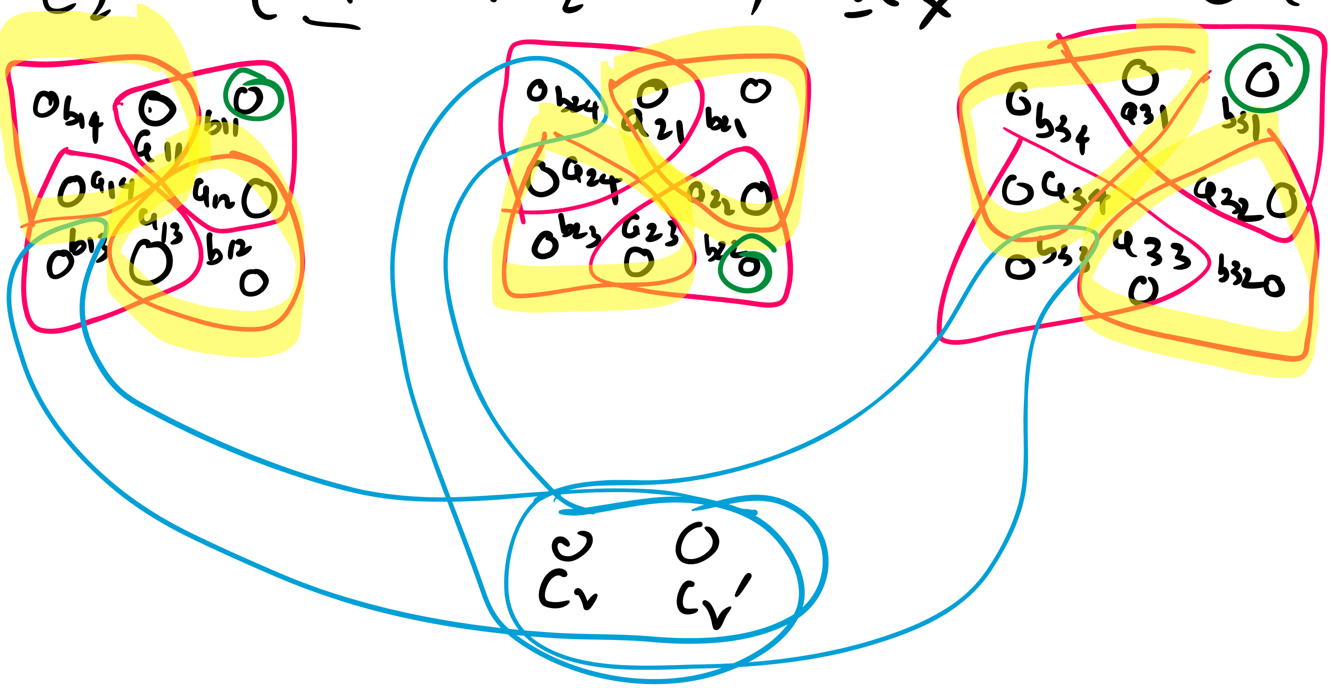
$l_j = x_j$
$\neg x_j$

$l_k = x_k$
$\neg x_k$

$\underline{x_1}, x_{\underline{2}}, \underline{x_3}$ 2 clauses

$C_1 = (x_1 \vee \neg x_2 \vee x_3)$

$C_2 = (x_1 \vee \neg x_2 \vee x_3)$



CLEANUP-GADGETS

\# tip-vertices $= 2nm$

\# tip-vertices covered by matching inside variable gadgets $= nm$

\# tip vertices covered by clause gadgets $= m$

\# tip vertices remain $= nm - m$
$= (n-1)m$

Introduce $(n-1)m$ cleanup gadjets

1) Add $cG_i, cG'_i \quad \forall i \in [(n-1)m]$

2) Add $(cG_i, cG'_i, b) \quad \forall i \in [(n-1)m]$
$\forall$ the tip

2) Add (CG_i, CG'_i, b) $\forall i \in [(n-1)m]$ $\forall b \in$ tip-vertex

Valid-reduction

I is Satisfiable $\iff$ I' admits a perf 3DM

$\Rightarrow$ A is a satisfying assignment

$x_i = T$, match var gadget of x_i ODDLY

$x_i = F$, - - - - - - - EVENLY

m tip-vertices covered by clause-gadgets (cover all clause vertices)

$(n-1)m$ tip-vert - - - - Cleanup gadgets (covers all cleanup vertex)

$\Leftarrow$ $\exists$ 3DM. T

to cover all core-vertices, T must match every variable gadget oddly or evenly.

Set $x_i = T$ if var gad x_i is matched ODDLY

$x_i = F$ if var gad; x_i is matched EVENLY

All clauses are satisfied by this assignment

3SAT $\leq_p$ 3DM

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SUBSET-SUM

Given :-> 1) set S of integers
2) Target t.

Question :-> $\exists S' \subseteq S$ such $\sum_{s \in S'} s = t$.

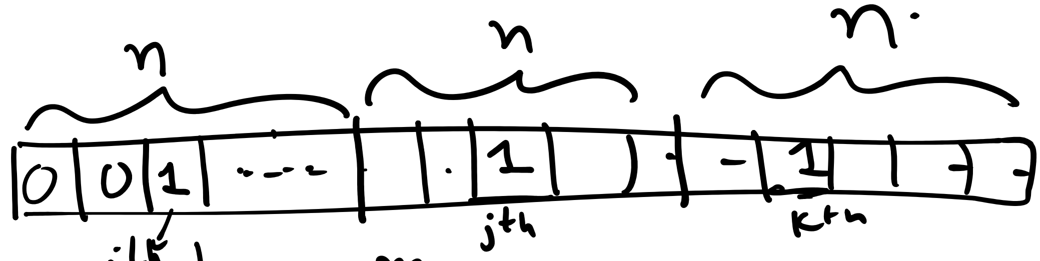
Thm :-> SUBSET-SUM $\leq_p$ NP-complete

1) SUBSET-SUM $\in$ NP (straightforward)

2) 3DM $\leq_p$ SUBSET-SUM.

↳ 1) X, Y, Z, $|X| = |Y| = |Z| = n$
2) set of triples $t_1, t_2, \ldots, t_m$

s



base-m

$t_1 = (x_i, y_j, z_k)$.
$\uparrow$ X, $\uparrow$ Y, $\uparrow$ Z

$$t = \sum_{i=0}^{3n} m^i = \frac{m^{3n} - 1}{m - 1}$$

t has a perfect 3DM $\Leftrightarrow$ $\exists S \subseteq S$ that

I has a perfect 3DM $\Leftrightarrow$ $\exists\, S \subseteq S$ that sums to t

$\Rightarrow$ $S = \bigcup_{i \in [n]} S_i$ where S_i = number corresponds to the triple t_i

$\bigcup_i t_i$ = 3DM in I

$\Leftarrow$ $\exists\, S' \subseteq S$ s.t $\underline{\sum_{s \in S'} s = t}$.

PARTITION: Given:→ 1) Set of integers S

Question:→ $\exists\, S' \subseteq S$ s.t $\sum_{s \in S'} s = \frac{1}{2}\left(\sum_{s \in S} s\right)$

Thm:→ PARTITION is NP-complete

1) PARTITION $\in$ NP

2) SUBSET-SUM $\leq_p$ PARTITION.

SUBSET-SUM → 1) S 2) target t

PARTITION: q = sum(S) → $S' = S \cup \{q\} \cup \{2t\}$

$\exists\, T \subseteq S$ s.t sum(T) = t $\Leftrightarrow$ $\exists\, T' \subseteq S'$ s.t

$\exists\ \underline{T \subseteq S}$ s.t $\underline{\text{sum}(T) = t}$ $\iff$ $\exists\ T' \subseteq S'$ s.t

$\text{sum}(T') = \underline{\text{sum}(S')/2}$

$\downarrow$

$\underbrace{\text{sum}(S) + \text{sum}(S) + 2t}_{2}$

$= \underline{\text{sum}(S) + t}$

$\Rightarrow$ $T \cup \{q\} \subseteq S'$

$\text{Sum}(T \cup \{q\}) = \text{sum}(T) + \text{sum}(q)$

$= t + \text{sum}(S)$

$= \text{sum}(S')/2$.

$\Leftarrow$ $\exists\ T' \subseteq S'$ $\text{sum}(T') = \text{sum}(S) + t$

T' contains atleast one of $\{q\}$ & $\{2t\}$

T' contains atmost one of $\{q\}$ & $\{2t\}$

$\Rightarrow$ T' contains exactly one of $\{q\}$ & $\{2t\}$

CASE-1 T' contains $\{q\}$

$\text{sum}(T') = \text{sum}(S) + t$

$\Rightarrow$ $\text{Sum}(\underline{T' \cap S}) + \cancel{\text{sum}(\{q\})} = \cancel{\text{sum}}(S) + t$

$\Rightarrow$ $T' \cap S \subseteq S$ s.t $\text{sum}(T' \cap S) = t$

CASE-2. T' contains $\{2t\}$

$$\mathrm{sum}(T') = \mathrm{sum}(S) + t$$

$$\Rightarrow \mathrm{sum}(T' \cap S) + 2t = \mathrm{sum}(S) + t.$$

$$\Rightarrow \mathrm{sum}(T' \cap S) + t = \mathrm{sum}(S)$$

$$\Rightarrow t = \mathrm{sum}(S \setminus (T' \cap S))$$

$$\Rightarrow t = \mathrm{sum}(\underline{S \setminus T'})$$

$S \setminus T' \subseteq S$ & $\underline{\mathrm{sum}(S \setminus T') = t}$

<u>KNAPSACK</u>.

<u>Given</u>:→ 1) Set S of items

2) u_i is the utility for item i

3) w_i is the volume for item i

4) Bag of size W.

<u>Goal</u>:→ Choose a set $S' \subseteq S$ of items that max $\sum_{i \in S'} u_i$

s.t. $\sum_{i \in S'} w_i \leq W$.

PARTITION $\leq_p$ KNAPSACK.
