Linear Programming

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Every $\underbrace{\text{LP } \Pi}_{\text{primal}}$ has a dual LP Π'

Motivation:

$$\max \; c^T x$$
$$Ax \le b$$
$$x \ge 0$$

. lower-bound :→ $c^T x'$ where x' is any feasible solution

upper-bound ?

Recall :→

$$\max \; 4x_2 + 3x_1$$
$$\text{s.t.} \quad 2x_2 + x_1 \le 3$$
$$x_1 + x_2 \le 2$$
$$x_1, x_2 \ge 0$$

$$4x_2 + 3x_1 = \underbrace{(2x_2 + x_1)}_{\le 3} + 2\underbrace{(x_1 + x_2)}_{\le 2}$$

$$= 3 + 2 \times 2 = 7$$

Constraints can give us an idea about the upper-bound!

ex-2.

$$\max \ 4x_1 + x_2 + 3x_3$$

s.t.

$$(x_1 + 4x_2 \le 1) \times y_1$$

$$+ \ (3x_1 - x_2 + x_3 \le 3) \times y_2$$

$$x_1, x_2, x_3 \geq 0$$

$$y_1 x_1 + 4x_2 y_1 \le y_1$$

$$+ \ 3y_2 x_1 - x_2 y_2 + y_2 x_3 \le 3y_2$$

$$(y_1 + 3y_2) x_1 + (4y_1 - y_2) x_2 + y_2 x_3 \le y_1 + 3y_2$$

$$y_1 + 3y_2 \approx 4$$

$$4y_1 - y_2 \approx 1$$

$$y_2 \approx 3$$

$y_1 + 3y_2$ is a tight upper-bond.

$$\min \ y_1 + 3y_2$$

s.t.

$$y_1 + 3y_2 \geq 4$$

$$4y_1 - y_2 \geq 1$$

$$y_2 \geq 3$$

$$y_1, y_2 \geq 0$$

DUAL LP

primal LP

$$\max\ c^T x \qquad \longrightarrow \qquad \text{DUAL}\quad \min\ b^T y$$

$$Ax \le b \qquad\qquad A^T y \ge c$$

$$x \ge 0 \qquad\qquad y \ge 0$$

Weak-Duality-Thm :- Let x be a feasible solution to the primal LP & y be a feasible solution to the Dual LP, then

$$c^T x \le b^T y$$

Pf :-

$$
\begin{aligned}
& c^T x \\
\le\ & (A^T y)^T x && (y \text{ is a feasible dual solution}) \\
=\ & y^T A x \\
\le\ & y^T b && (x \text{ is a feasible primal solution}) \\
=\ & b^T y
\end{aligned}
$$

Strong-Duality Thm :- If y^* and x^* are optimum solutions to the dual LP & the primal LP respectively, then

$$\boxed{c^T x^* = b^T y^*}$$

$\max c^T x$

$Ax \le b$

$x \ge 0$

OPTIMIZATION PBM

$\rightsquigarrow$ Strong Duality Thm

$Ax \le b$

$A^T y \ge c$

$b^T y = c^T x$

$x, y \ge 0$

FEASIBILITY CHECK

Applications :→

1) Shortest path LP
2) Max-flows LP.

Shortest paths :→ Given s & t; $w(u\to v) \ge 0$

$\underline{\underline{d_v}}$ $\forall\, v \in V$.

$\min b^T y$

$A^T y \ge c$

$y \ge 0$

$\leftarrow$

$\max c^T x$

$Ax \le b$

$x \ge 0$

LP for Shortest path

$\downarrow$

Dual LP?

$y_{(u\to v)}$

$$\max \; d_t$$

$$d_v - d_u \le w_{u\to v} \quad \forall\, (u\to v) \in E$$

$$d_x \ge 0 \quad \forall\, x \in V.$$

$\downarrow$

$$\min \sum_{(u\to v)\in E} y_{u\to v} \cdot w_{u\to v}$$

$$\text{s.t.} \sum_{\substack{(u\to v)\\ \in E}} y_{u\to v} - \sum_{\substack{(v\to u)\\ \in E}} y_{v\to u} \ge 0 \quad \forall\, v \in V \setminus t$$

$$\sum_{\substack{(u\to t)\\ \in E}} y_{u\to t} - \sum_{\substack{(t\to u)\\ \in E}} y_{t\to u} \ge 1$$

$$\sum_{(u\to t)\in E} y_{u\to t} - \sum_{(t\to u)\in E} y_{t\to u} \geq 1 \;/$$

$$y_{(v\to u)} \geq 0 \quad \forall\, (v\to u)\in E$$

Max-flows :→ **LP** Given flow network, s, t.

$$f_{u\to v} \quad \forall\, (u\to v)\in E \,.$$

$$\max \sum_{(s\to v)\in E} f_{s\to v}$$

Dual variables

$y_{u\to v}$	$f_{u\to v} \leq c_{u\to v}$
y_v	$\sum_{(u\to v)\in E} f_{u\to v} - \sum_{(v\to w)\in E} f_{v\to w} \leq 0 \quad \forall v \neq s,t$
y'_v	$-\sum_{(u\to v)\in E} f_{u\to v} + \sum_{(v\to w)\in E} f_{v\to w} \leq 0$

$$f_{u\to v} \geq 0$$

Dual LP

$$\min \sum_{(u\to v)\in E} c_{u\to v}\cdot y_{u\to v}$$

$$y_{s\to v} + y_v - y'_v \geq 1 \quad \forall\, (s\to v)\in E \,.$$

$$y_{s\to v} + y_v - y'_v \geq 1$$

$$y_{u\to v} + y_v - y'_v - y_u + y'_u \geq 0 \quad \forall (u\to v) \in G \setminus \{s\}$$

$$y_{v\to t} - y_v + y'_v \geq 0 \quad \forall (v\to t) \in E$$

$$y_{u\to v} \geq 0$$

$$y_v \geq 0, \quad y'_v \geq 0$$

$$d_v = y_v - y'_v$$

$$d_s = 0, \quad d_t = 1$$

After substitution

$$\min \sum_{(u\to v)\in E} c_{u\to v}\, y_{u\to v}$$

s.t. $d_v - d_u \leq y_{u\to v}$

$$y_{u\to v} \geq 0$$

$$d_s = 0, \quad d_t = 1$$

Claim :- optimum value of Dual LP = VAL_{MC}

S-1 $\quad OPT_D \leq VAL_{MC}$ $\quad (S, T)$ is the min-cut

$s \in S$, $t \in T$

$$d_v = 0 \quad \forall v \in S$$

$d_v = 1 \ \forall \ v \in T$

$y_{u \to v} = 1$ if $u \in S$ & $v \in T$

$= 0$ otherwise.

$$VAL_{MC} = \sum_{\substack{u \in S \\ v \in T}} c_{u \to v} \cdot \underline{y_{u \to v}} \geq OPT$$

$OPT \geq VAL_{MC}$

Let d^* & y^* are the optimum solution

Pick $\underline{z} \in (0, 1)$ uniformly at random

$$\underset{s}{S} = \{ u \mid d^*_u \leq z \} \quad \underset{t}{T} = \{ u \mid d^*_u > z \}$$

(S, T) is a valid-cut

$$VAL_{MC} \leq E[\text{capacity of the cut } (S, T)]$$

$$= \sum_{\substack{E_{u \to v} \in E \\ u \in S \\ v \in T}} c_{u \to v} \Pr[u \in S \ \& \ v \in T]$$

$$= \sum c_{u \to v} \max(0, d^*_v - d^*_u)$$

$$\leq \sum c_{u \to v} \, y^*_{u \to v}$$

$\leq \sum c_{u\to v}\, y^*_{u\to v}$

$= OPT$

$VAL_{MC} = OPT$

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max-flow LP $\xrightarrow[\text{pr}]{\text{dual}}$ Min-cut

Strong-duality $\Rightarrow$ max-flow-min-cut thm