Tuesday, April 4, 2023 1:51 PM

Linear Programming

$$\max_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n)$$

$$h_i(x_1, x_2, \ldots, x_n) \leq 0 \quad \forall i \in [n]$$

Kantorovich 1939

e.g.

$$\max \ 4x_2 + 3x_1 \quad \} \rightarrow \text{objective fn}$$

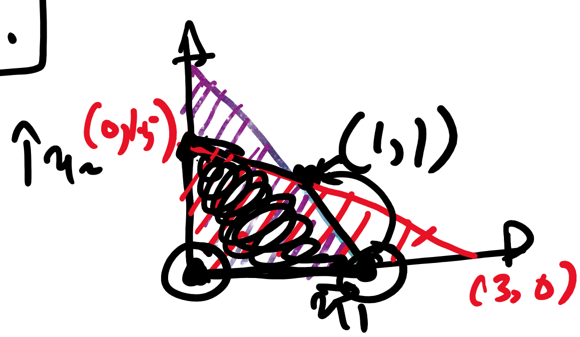
s.t.
$$\left.\begin{aligned} 2x_2 + x_1 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}\right\} \text{Polytope}$$

$$4x_2 + 3x_1 = \underbrace{(2x_2 + x_1)}_{\leq 3} + 2\underbrace{(x_1 + x_2)}_{4} = \underline{\underline{7}}$$

$$\boxed{x_1 = x_2 = 1.}$$

$2x_2 + x_1 \leq 3$

$x_1 + x_2 \leq 2$



Applications :-

Max-flow
(s, t)

$x_{u \to v} \quad \forall (u \to v) \in E$

$$\max \sum_{(s \to v) \in E} x_{s \to v}.$$

$x_{u \to v} \geq 0.$

$x_{u \to v} \leq c_{u \to v}$

$$\sum_{(u \to v) \in E} x_{u \to v} \;\substack{\geq \\ \leq}\; \sum_{(v \to u) \in E} x_{v \to u} \qquad \forall v \in V \setminus \{s, t\}$$

Linear Program

$$\Pi = \left\{ \boxed{\begin{array}{l} \max \sum_{j \in [n]} c_j x_j \\ \sum_{j \in [n]} a_{ij} x_j \leq b_i \end{array}} \right. \quad \forall i \in [m]$$

Replace every x_j by $y_j - z_j$

$$\Pi' : \left\{ \begin{array}{l} \max \sum_{j \in [n]} c_j (y_j - z_j) \\ \text{s.t.} \quad \sum_{j \in [n]} a_{ij} (y_j - z_j) \leq b_i \quad \forall i \in [m] \\ y_j, z_j \geq 0. \end{array} \right.$$

Standard Form :→ $\max c^T x$

s.t. $Ax \leq b$

s.t $Ax \le b$
$x \ge 0$

SIMPLEX ALGORITHM

$$\max \sum_j c_j x_j$$

s.t. $\sum_j a_{ij} x_j \le b_i$

$x_j \ge 0$

$\longrightarrow$

$$\max \sum_j c_j x_j$$
$$\sum_j a_{ij} x_j + s_i = b_i$$
$$x_j \ge 0 \;\; \forall j \in [n]$$
$$s_i \ge 0 \;\; \forall i \in [m]$$

Slack form

$$\max c^T x$$
$$Ax = b$$
$$x \ge 0$$

n = |Non-basic variables|

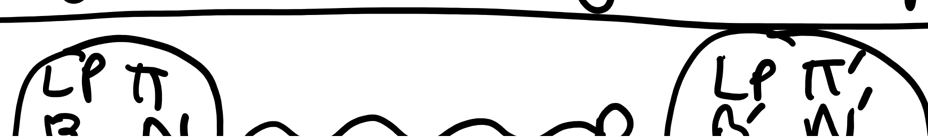
m = |basic variables|

Basic-Solution :→ A solution where all non-basic variables are set to zero.

Feasible-Solution :→ A solution which satisfies the system of equalities & non-negativity

Optimal solution :→ A feasible solution with the maximum value of the OBJ function

High-level-Structure of the Simplex Alg



LP Π
B, N
BFS x

"pivoting"

$\Pi \equiv \Pi'$

LP Π'
B', N'
BFS x'

$$B' = B \setminus \underset{\substack{\Uparrow \\ B}}{\underline{\{x_i\}}} \cup \underset{\substack{\Uparrow \\ N}}{\{x_j\}}$$

$$N' = N \setminus \{x_j\} \cup \{x_i\}$$

$$|B'| = |B|, \qquad |N'| = |N|$$

$$OBJ(x') > OBJ(x)$$

PIVOTING

$$\max \underline{v} + \sum_j c_j x_j$$

$$\text{s.t.} \quad s_i = b_i - \sum_j a_{ij} x_j \quad \forall i$$

BV

$$s_i \geq 0 \ \forall i, \quad x_j \geq 0 \ \forall j$$

$\exists e$ s.t. $c_e > 0 \Rightarrow$ increase x_e.

$$s_i = b_i - a_{ie} x_e \geq 0 \Rightarrow x_e \leq b_i / a_{ie}$$

x'

Set $x_e = \min\limits_{\substack{i \in [m] \\ \text{s.t. } a_{ie} > 0}} b_i / a_{ie}$:

Let i^* be $\operatorname{argmin}\limits_{i \in [m]} b_i / a_{ie}$, set $\begin{cases} s_{i^*} = 0 \\ s_i = b_i - a_{ie} x_e \end{cases}$

$B' = B \setminus \{s_{i^*}\} \cup \{x_e\}$

$$\begin{cases} B' = B \setminus \{s_{i^*}\} \cup \{x_e\} \\ N' = N \setminus \{x_e\} \cup \{s_{i^*}\} \end{cases}$$

update π to π'

$$s_{i^*} = b_{i^*} - \sum_j a_{i^*j} x_j$$

$$\Rightarrow \quad x_e = \frac{b_{i^*}}{a_{ie}} - \sum_{j \neq e} \frac{a_{i^*j}}{a_{ie}} x_j - \frac{s_{i^*}}{a_{ie}} \quad (*)$$

Replace every occurrence of x_e in the Objective function and all OTHER constraints by $(*)$

Moved from $\begin{matrix} \text{LP } \pi \\ B, N \\ \text{BFS } x \end{matrix} \rightsquigarrow \begin{matrix} \text{LP } \pi' \\ B', N' \\ \text{BFS } x' \end{matrix}$

$$OBJ(x') > OBJ(x)$$

Running time: $\to \binom{n+m}{m}$

Initialization: $\to$

$\min x_0$

$$S_i = b_i - \sum_j a_{ij} x_j + x_0$$

$S_i \geq 0 \quad \forall i \in [m]$

$x_j \geq 0 \quad \forall j \in [n]$

$x_0 \geq 0$

Start with $x_0 = L$ (arbitrarily large)

$L > \max_{i \in [m]} |b_i|$

Fundamental Thm of Linear Programming

Given any LP Π,

1) If no optimum soln exists, then Π is either infeasible or unbounded

2) If $\exists$ feasible solution, then there exists a basic feasible soln.

3) If $\exists$ optimum solution, there exist a basic optimum solution

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1) Ellipsoid Alg $\to$ First Poly-time alg for LP
Kachian 1979

2) Interior Point Alg $\to$ Karmakar 1984.