$\forall u \in S, v \in T,\quad (u,v) \in E \Rightarrow f(u,v) = c(u,v)$
$\qquad\qquad\qquad\qquad (v,u) \in E \Rightarrow f(u,v) = 0$

$$\text{max flow value} \geq |f| = \sum_{\substack{u \in S \\ v \in T}} f(u,v) = \text{capacity of } (S,T)$$
$$\geq \text{capacity of min cut}$$
$$\geq \text{max flow value by obs}$$

$\square$

Runtime Analysis:

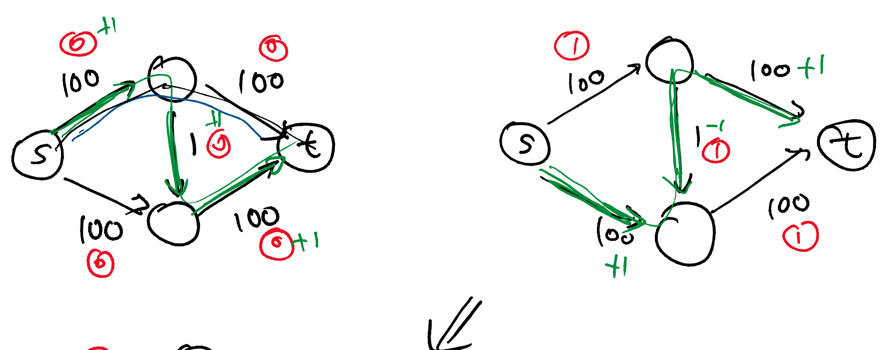
each iteration takes $O(m)$ time by DFS/BFS

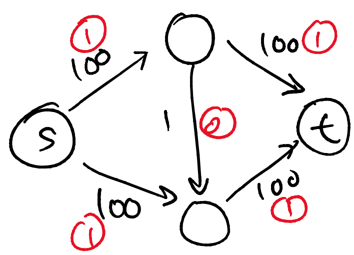
if capacities are integers in $\{1, \ldots, U\}$,

$\#$ iterations $\leq |f^*| \leq \underline{nU}$

$\Rightarrow \boxed{O(mnU)}$ total time

Bad Ex





$\sim \underline{2}00$ iterations

if not integers, may not even terminate ...

Integrality Thm If all input capacities are integers, opt flow has integer values.

Pf: By Ford-Fulkerson. $\square$

Edmonds-Karp Version 1 ('72)

always pick aug. path with largest bottleneck value Δ

modified Dijkstra
to find st bottleneck path
$O(m \log n)$ time

Analysis:

Claim # iterations $\leq O(m \log |f^*|)$
$\leq O(m \log (nU))$

Pf Sketch:

Fact (Flow Decomposition Thm)
Any flow can be expressed as a sum of $\leq m$ cycle flows & path flows.

Let f_i^* be max flow in G_f in i^{th} iteration.
By Decomp. Thm, Can decompose f_i^* into $\leq m$ cycle flows & path flows in G_f. remove

$\Rightarrow \exists$ path flow in G_f with $\geq \frac{|f_i^*|}{m}$ value
augmenting path in G

$$\Rightarrow |f_{i+1}^*| \leq |f_i^*| - \frac{|f_i^*|}{m}$$
$$= \left(\frac{m-1}{m}\right)|f_i^*|.$$

$$\Rightarrow \text{\# iterations} \leq \log_{\frac{m}{m-1}} |f^*|$$
$$= O\left(\frac{\log |f^*|}{\log\left(1+\frac{1}{m-1}\right)}\right)$$
$$= O(m \log |f^*|). \quad \square$$

$\log(1+x) \sim x$

$$\Rightarrow \text{total time } O(m \log n \cdot m \log |f^*|)$$
$$= O(m^2 \log n \log(nU))$$

$$= \boxed{O(m^2 \log n \log(nU))}$$

Edmonds-Karp Version 2:

always pick shortest aug path
i.e. fewest hops

each iterations $O(m)$ time by BFS

Claim # iterations $\le mn$.
Pf omitted.

$\Rightarrow$ total time $\boxed{O(m^2 n)}$

Later History:

Dinic '70	$O(mn^2)$
Goldberg-Tarjan '86	$O(mn \log n)$.
Goldberg-Rao '98	$O(m^{3/2} \log^2(nU))$
	$O(m n^{2/3} \log^2(nU))$
⋮	
Lee-Sidford '15	$O(m\sqrt{n} \log^c(nU))$
Madry '16	$\tilde{O}(m^{10/7} U^{1/7})$
'20	$\tilde{\tilde{O}}(m^{4/3} U^{1/3})$
'20	$\tilde{O}((m + n^{3/2}) \log U)$
'21	$\tilde{O}(m^{\frac{3}{2} - \frac{1}{328}} \log U)$
'22	$\tilde{O}(m^{\frac{3}{2} - \frac{1}{58}} \log U)$
'22	$O(m^{1+o(1)} \log U)$

$\ll \qquad O(m \cdots \log U)$

(> 112 pp)

Generalization: Min-Cost Flow

Given cost: $E \to \mathbb{R}^+$ and d,
find flow f with value $|f| = d$
minimizing $\displaystyle\sum_{(u,v)\in E} \text{cost}(u,v) \cdot f(u,v)$.

Appl: min-cost perfect matching, ...

Define residual graph G_f as before;
for $(v,u) \in E$, define $\text{cost}(u,v) = -\text{cost}(v,u)$.

Alg'm (Klein '67) by "Cycle Canceling"

f = any flow with value d (by max flow alg'm)
repeat {
find a negative-cost cycle γ in G_f
by DP (Bellman-Ford) $O(mn)$ time
Δ = bottleneck value in γ
$\forall (u,v) \in \gamma$, $f(u,v) \mathrel{+}= \Delta$
$f(v,u) \mathrel{-}= \Delta$.
}

Correctness: flow f has min-cost
$\iff$ $\nexists$ no neg-cost cycle in G_f.

... capacities are integers in $\{1,..,U\}$, ...

$\Leftrightarrow \nexists$ no ...

Analysis: if capacities are integers in $\{1, .., U\}$,
costs are integers in $\{1, .., C\}$,

\# iterations $\leq$ cost(f^*) $\leq$ $m \cdot U \cdot C$

$\Rightarrow$ total time $O(mn \cdot mUC) = \boxed{O(m^2 n U C)}$

⋮

Midterm 2: Apr 3 Mon 7p - 9p (Loomis 141)

Opt. Review on Thurs.