\[ \forall u \in S, v \in T, (u,v) \in E \Rightarrow f(u,v) = c(u,v) \]
\[ (v,u) \in E \Rightarrow f(u,v) = 0 \]

**Runtime Analysis:**

Each iteration takes \( O(m) \) time by DFS/BFS.

If capacities are integers in \( \{1, \ldots, U\} \),

\[ \# \text{ iterations} \leq \left\lceil \frac{1}{U} \right\rceil \leq nU \]

\[ \Rightarrow O(mnU) \text{ total time} \]

**Bad Ex**

If not integers, may not even terminate ...

**Integrality Thm**

If all input capacities are integers, opt flow has integer values.

**PF:** By Ford-Fulkerson.
Edmonds-Karp Version 1 ('72)

always pick aug. path with largest bottleneck value Δ

modified Dijkstra to find bottleneck path

O(m log n) time

Analysis:

Claim: # iterations \leq O\left(\frac{m \log f^*}{m}\right) \leq O\left(m \log(mU)\right)

Pf Sketch:

Fact (Flow Decomposition Thm)
Any flow can be expressed as a sum of
≤m cycle flows & path flows.

Decompose f^* into ≤m cycle flows & path flows in G^p.

⇒ ∃ path flow in G with \geq \frac{|f^*|}{m} value augmenting path in G

⇒ |f^{k+1}| \leq |f^k| - \frac{|f^*|}{m} = \frac{m-1}{m} |f^*|.

⇒ # iterations \leq \log \frac{m}{m-1} |f^*|

\log(\text{iter}) \sim m

O\left(\frac{\log f^*}{\log (m \frac{1}{m-1})}\right)

= O\left(\frac{\log f^*}{m \log f^*}\right)

= O\left( m \log f^*\right).

⇒ total time \ O(m \log n \cdot m \log f^*)

= O\left(\frac{m^2 \log n \log(mU)}{m}\right)
Edmonds-Karp Version 2:

always pick shortest aug path
  i.e. fewest hops

each iterations \(O(m)\) time by BFS

\(\text{Claim} \ # \text{iterations} \leq mn.\)

\(\text{Pf omitted.}\)

\[=) \text{Total time } O(m^2n)\]

Later History:

- Dvij '70  \(O(mn^2)\)
- Goldberg-Tarjan '86 \(O(mn \log n)\)
- Goldberg-Rao '98
  \[O(m^{3/2} \log^2 (nU))\]
  \[O(m n^{2/3} \log^2 (nU))\]
  \[O(m n^{3/4} \log^2 (nU))\]
- Lee-Sidford '15
  \(O(m n \log (\sqrt{nU}))\)
- Madry '16
  \(\tilde{O}(m^{11/17} \sqrt{nU})\)

'120 \(\tilde{\Omega}(m^{4/3} \sqrt{U})\)

'120 \(\tilde{O}((m + n^{3/2}) \log U)\)

'121 \(\tilde{O}(m^{2 - \frac{3}{28}} \log U)\)

'122 \(\tilde{O}(m^{3/5} \log U)\)

'122 \(O(m^{1 + o(1)} \log U)\)
Generalization: Min-Cost Flow

Given cost: $E \rightarrow \mathbb{R}^+$ and $d$,
find flow $f$ with value $|f| = d$
minimizing $\sum_{(u,v) \in E} \text{cost}(u,v) \cdot f(u,v)$.

Appl: min-cost perfect matching,

Define residual graph $G_f$ as before:
for $(u,v) \in E$, define $\text{cost}(u,v) = -\text{cost}(v,u)$.

Algm (Klein '67) by "Cycle Canceling"

$f =$ any flow with value $d$ (by max flow algm)
repeat {
find a negative-cost cycle in $G_f$
by DP (Bellman-Ford)
$O(mn)$ time

$\Delta =$ bottleneck value in $\gamma$
for $(u,v) \in \gamma$, $f(u,v) += \Delta$
$\quad f(v,u) -= \Delta$.

}$

Correctness: flow $f$ has min-cost
$\iff$ no neg-cost cycle in $G_f$.
Analysis: if capacities are integers in \{1, \ldots, U\},
costs are integers in \{1, \ldots, C\},

\[ \text{# iterations} \leq \text{cost}(f^*) \leq m U C \]

\[ \Rightarrow \ O(mn, mUC) = O(m^2 U C) \]

Midterm 2: Apr 3 Mon 7p-9p (Loomis 141)