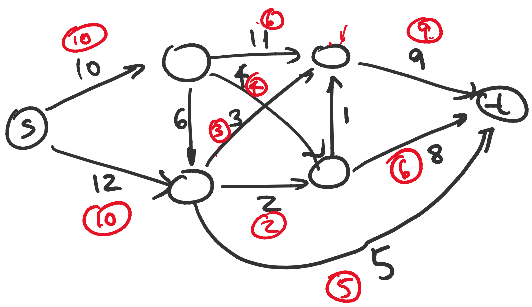
March 23, 2023 1:46 PM

Max Flow

Problem Given directed graph ("network") $G = (V, E)$, $s, t \in V$ (source (in-deg $=0$), sink (out-deg $=0$))
$|V| = n$, $|E| = m$,
where each $e \in E$ has a capacity $c(e) > 0$,
find flow from s to t with max value.



value 20

Def A flow is $f: E \to \mathbb{R}$ s.t.

$\forall (u,v) \in E, \quad 0 \le f(u,v) \le c(u,v)$

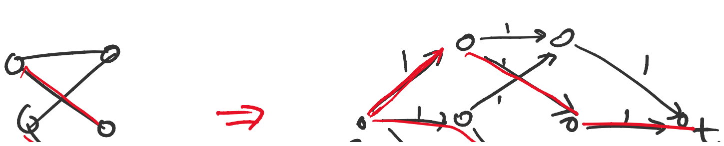
(flow conservation) $\forall v \in V \setminus \{s,t\}: \quad \sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$

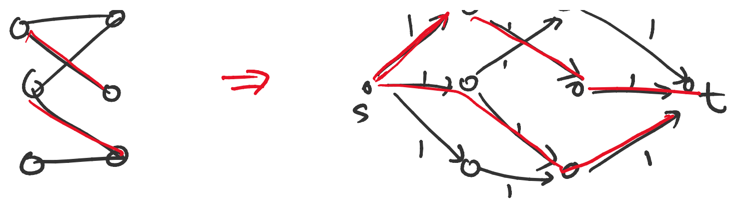
Value of $f = |f| = \sum_{v:(s,v)\in E} f(s,v)$

$\left(= \sum_{v:(v,t)\in E} f(v,t) \right)$

(**Convention:** make $f(v,u) = -f(u,v)$ if $(u,v) \in E$
& $f(u,v) = 0$ if $(u,v), (v,u) \notin E$.
then flow conservation means $\sum_{w \in V} f(v,w) = 0$.)

Appl / Special Case 1: max bipartite matching





(turns out: opt flow is integral)

Appl/ Special Case 2: find max # of edge-disjoint paths from s to t in dir. graph.

(vertex-disjoint undir.) . . .

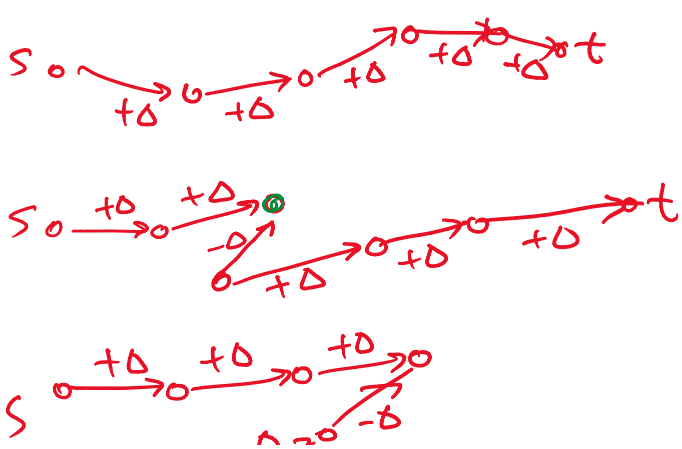
Appl/ Special Case 3.: assignment problems

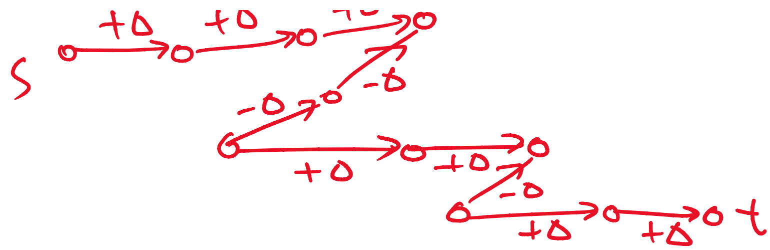
assign jobs to people
each person u can be $\leq c_u$ jobs
max # assignments

Etc.

Ford & Fulkerson's Alg'm ('56)

idea - local improvement again
augmenting path again





Def Let f be a flow.

Define G_f : residual graph

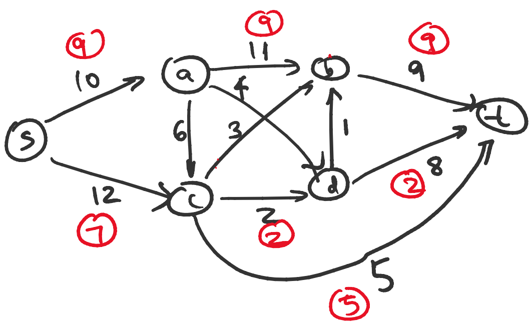
if $(u,v) \in E$, $f(u,v) < c(u,v)$, not saturated

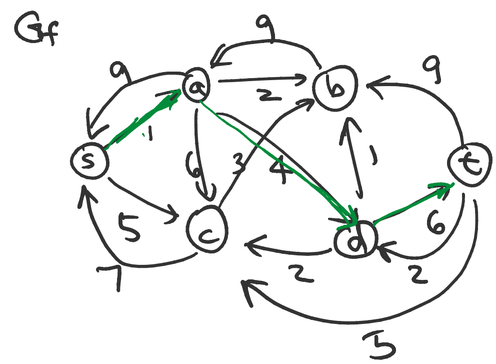
add (u,v) in G_f with capacity $c_f(u,v) = c(u,v) - f(u,v)$

if $(v,u) \in E$, $f(v,u) > 0$,

add (u,v) in G_f with capacity $c_f(u,v) = f(v,u)$

Ex





Algm:

f = zero flow

repeat {

find an s-to-t path p in G_f called augmenting path

if not exist, return f

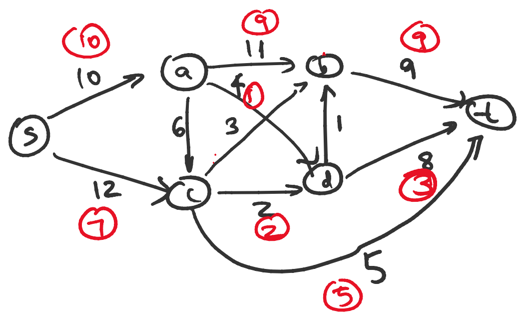
Δ = min edge capacity in p in G_f called bottleneck value

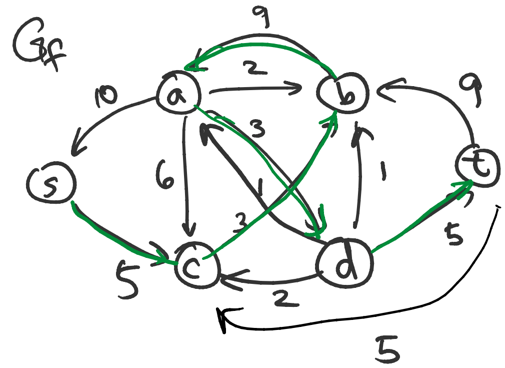
$\forall (u,v) \in p$, $f(u,v) \mathrel{+}= \Delta$

$f(v,u) \mathrel{-}= \Delta$.

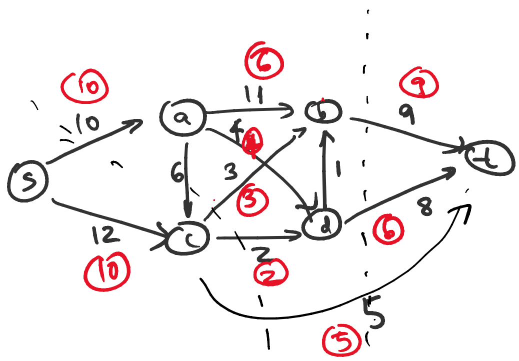
}

Ex $\Delta = 1$





$\Delta = 3$:



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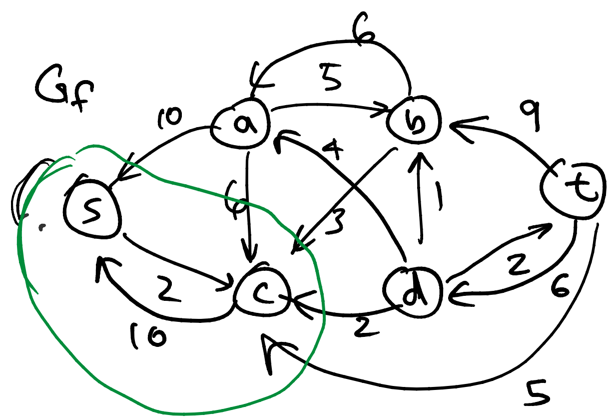


no path from s to t

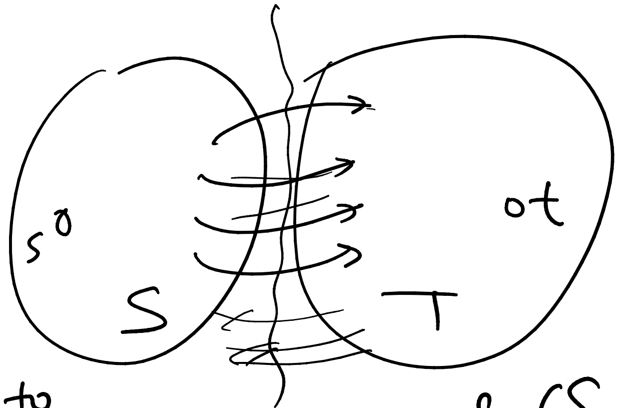
Correctness:

Thm 1 flow f is optimal $\iff$ it has no augmenting path.

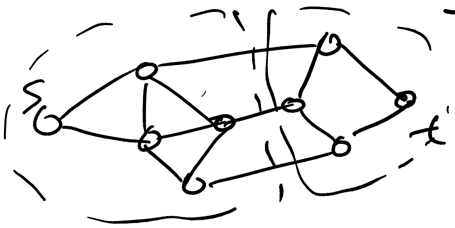
Pf: how?

Min s-t Cut

Def An <u>s-t cut</u> is a partition (S, T) of V with $s \in S$, $t \in T$



Want to minimize the capacity of (S,T)



$$= \sum_{\substack{u\in S, v\in T \\ (u,v)\in E}} c(u,v)$$

__Obs__ max flow value $\le$ min-(s,t)-cut capacity.

Pf: $$|f| = \sum_{u\in S, v\in T} f(u,v) \le \sum_{\substack{u\in S, v\in T\\(u,v)\in E}} c(u,v) - \cancel{\sum_{\substack{u\in S, v\in T\\(v,u)\in E}} 0}$$

$$= \text{capacity of } (S,T). \quad \square$$

__Thm 2__ max flow value $=$ min-(s,t)-cut capacity

__Pf of Both Thms :__

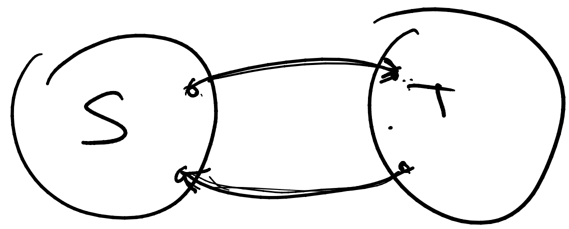
Let f be flow with no aug path.

Look of G_f.

s & t are not reachable in G_f

Let S = all vertices reachable from s in G_f

$T = V - S$.



$\forall u \in S, v \in T,\quad (u,v) \in E \Rightarrow f(u,v) = c(u,v)$

$\qquad\qquad\qquad\quad (v,u) \in E \Rightarrow f(u,v) = 0$

$$\text{max flow value} \geq |f| = \sum_{\substack{u \in S \\ v \in T}} f(u,v) = \text{capacity of } (S,T)$$

$$\geq \text{capacity of min cut}$$

$$\geq \text{max flow value} \quad \text{by Obs}$$

$\square$

Runtime Analysis:

each iteration takes $O(m)$ time by DFS/BFS

if capacities are integers in $\{1, \ldots, U\}$,

\# iterations $\leq |f^*| \leq nU$

$\Rightarrow \boxed{O(mnU)}$ total time