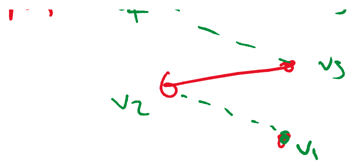


$S \oplus T$
$= S \cup T - (S \cap T)$

$\Rightarrow$ $M \oplus P$ is a matching with $|M \oplus P| \geq |M| + 1$.

Algm:

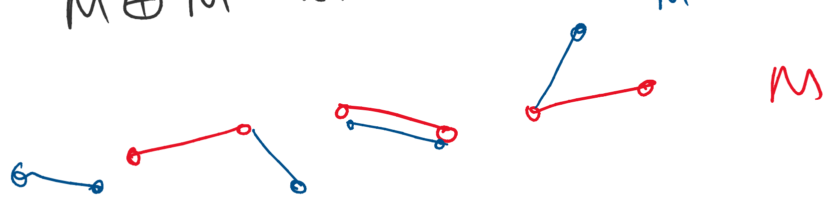
$M = \emptyset$
repeat {
  find augmenting path P for M
  if not exist, return M
  $M = M \oplus P$
}

Thm: matching M is optimal $\iff$ it has no augmenting path.

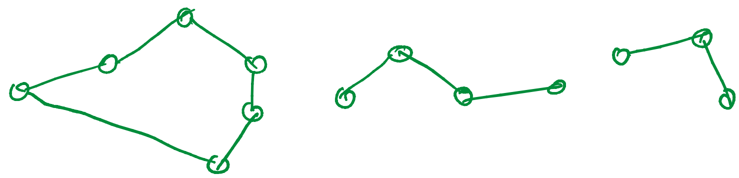
each iteration: $O(m)$ time
iterations: $\leq n$
$\Rightarrow \boxed{O(mn)}$

Lemma If M and M^* are matchings with $|M| < |M^*|$, then $M \oplus M^*$ contains $\geq |M^*| - |M|$ vertex-disjoint augmenting paths for M.

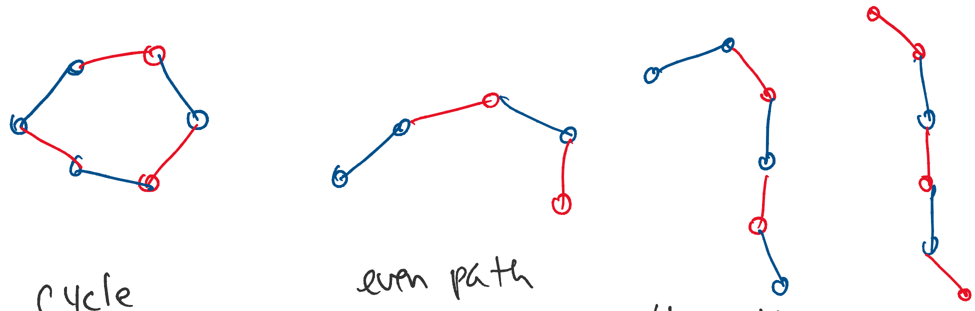
Pf: $M \oplus M^*$ has max deg ≤ 2.



Fact Any undir. graph with max deg ≤ 2 can be decomposed into vertex-disjoint paths & cycles.



$M \oplus M^*$



$$\underbrace{|M^*|}_{\#\text{ blue edges}} - \underbrace{|M|}_{\#\text{ red edges}} = \#\text{ blue-odd paths} - \#\text{ red-odd paths}$$

$$\leq \underbrace{\#\text{ blue-odd paths}}_{\text{these are aug paths}}$$

$\square$

Pf of Thm:

if $\exists$ aug path, then M is not optimal.

if M is not optimal,

by Lemma, $\exists \geq 1$ aug path. $\square$

How to find an augmenting path?

form directed graph G_M: $\forall$ u on left side, v on right side,

O(m) time

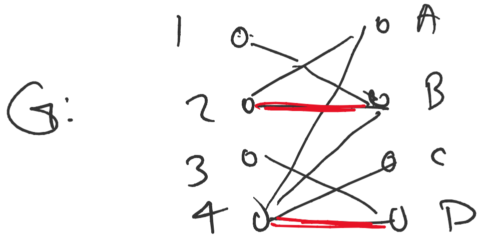
- if $uv \in E - M$, add (u,v) to G_M.
- if $uv \in M$, add (v,u) to G_M.
- if u not covered by M, add (s,u) to G_M
- if v not covered by M, add (v,t) to G_M

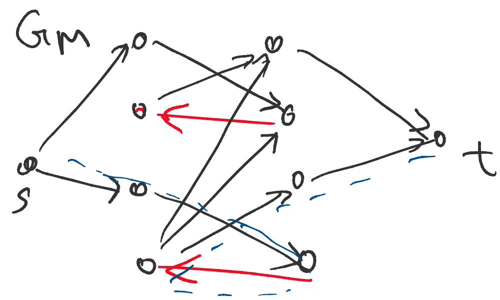
O(m) time

if u not covered by M, ...

if v not covered by M, add (v, t) to G_M

run BFS / DFS on a directed graph G_M from s to t





$M = \{2B, 4D\}$

iterations $\leq |M^*|$ since $|M|$ increases by 1 per iteration
$\leq n$

$\Rightarrow \boxed{O(mn)}$ time

Hoperoft & Karp's Alg/m '73: Sketch

idea 1 - always pick shortest augmenting path

idea 2 - augment multiple vertex-disjoint shortest paths per iteration. ←

Fact length of shortest paths increases per iteration

$\Rightarrow$ after $\sqrt{n}$ iterations, path lengths $\geq \sqrt{n}$

$\Rightarrow$ $\leq \sqrt{n}$ vertex-disj. aug paths

$\Rightarrow$ by Lemma, $|M^*| - |M| \leq \sqrt{n}$.

$\Rightarrow$ $|M| \geq |M^*| - \sqrt{n}$

$\Rightarrow$ # remaining iterations $\leq \sqrt{n}$.

$\Rightarrow \boxed{O(m\sqrt{n})}$ time

Extensions: unweighted general (non-bipartite)

Edmonds '65 $O(mn^2)$

$\downarrow$

'80 $O(m\sqrt{n})$

weighted bipartite

weighted non-bipartite

$\vdots$

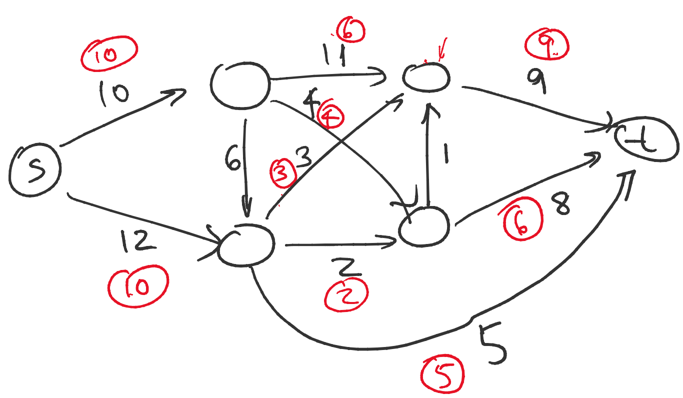
Max Flow

Problem Given directed graph $G=(V,E)$, $s,t \in V$ ("network"; s: source (in-deg = 0); t: sink (out-deg = 0))

$|V|=n$, $|E|=m$,

where each $e \in E$ has a capacity $c(e) > 0$,

find flow from s to t with max value.



value 20

Def A flow is $f: E \to \mathbb{R}$ s.t.

$\forall (u,v) \in E$, $0 \le f(u,v) \le c(u,v)$

flow conservation $\forall v \in V - \{s,t\}$: $\displaystyle\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$

Value of f = $|f| = \displaystyle\sum_{v:(s,v)\in E} f(s,v)$

$\left(= \displaystyle\sum_{v:(v,t)\in E} f(v,t) \right)$

(Convention: make $f(v,u) = -f(u,v)$ if $(u,v) \in E$
& $f(u,v) = 0$ if $(u,v),(v,u) \notin E$.
then flow conservation means $\displaystyle\sum_{w\in V} f(v,w) = 0$.)

Apply Special Case 1: max bipartite matching