$$= E\left[\sum_{y \in S} O\left(\frac{n}{m}\right)\right]$$

$$= O\left(n \cdot \frac{n}{m}\right).$$

Set $m = cn^2$ $\Rightarrow$ $E[\text{total \# colliding pairs}] = O\left(\frac{1}{c}\right)$

$\Rightarrow$ $\Pr[\text{total \# colliding pairs} \geq 1] \leq O\left(\frac{1}{c}\right)$.

by Markov's ineq

$\Rightarrow$ $O(n^2)$ space
$O(1)$ worst-case query time

perfect hashing

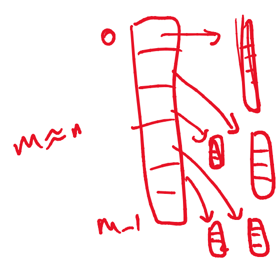
$O(n)$ expected preproc time
by repeating $O(c)$ expected # times..

Can space be reduced back to $O(n)$?

yes ... to be cont'd ...

Final Method

idea - bootstrap



for each bucket $A[i]$,
store it in perfect hash table
with $O(|A[i]|^2)$ space
$O(1)$ worst-case query time

Analysis:

expected space $= O\left(m + E\left[\sum_{i=0}^{m-1} |A[i]|^2\right]\right)$

expected space $= O\left(m + E\left[\sum_{i=0} |A[i]|^2\right]\right)$

$= O\left(m + E[\#\text{colliding pairs}]\right)$ $\qquad \sum_{i=0}^{m-1} \binom{|A[i]|}{2}$

$= O\left(m + \frac{n^2}{m}\right)$

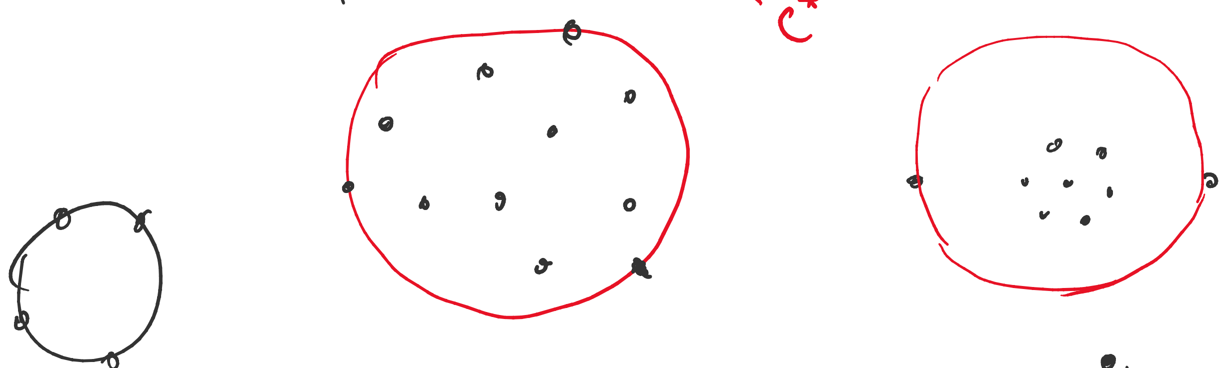
pick $m \simeq n$ $\quad = \boxed{O(n)}$.

$\Rightarrow$

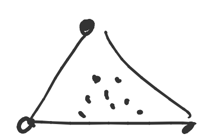
$O(1)$	worst-case query time
$O(n)$	~~expected~~ worst-case space
$O(n)$	expected preproc. time

Rmks. other universal families of hash fns

Problem Given set S of n points in 2D, compute the min circle enclosing S. $\leftarrow C^*$



appl. - bounding objects ...
- facility location ......

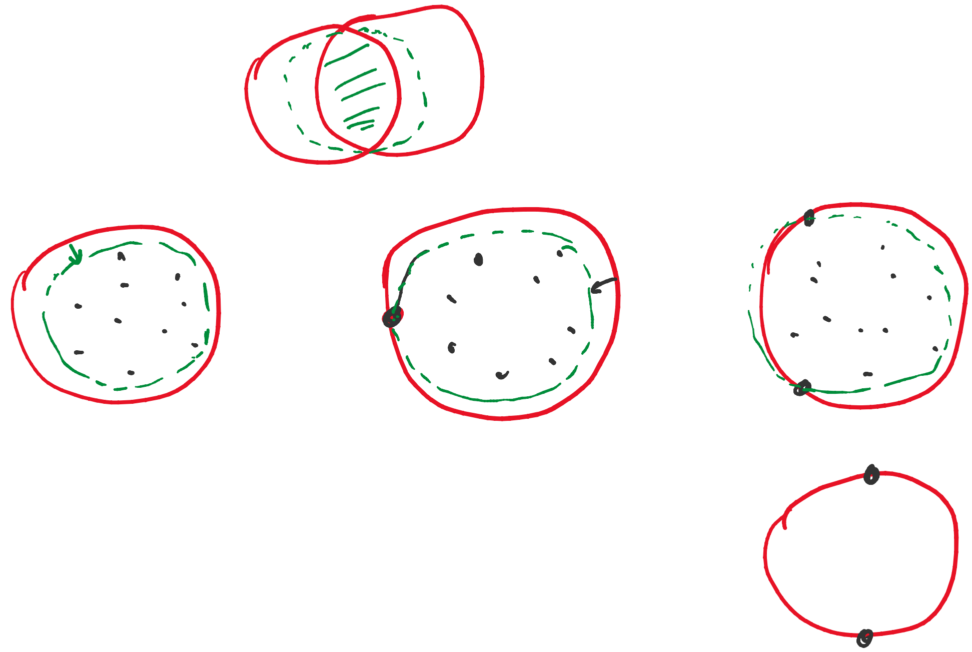


Obs C^* is unique & must have ≥ 3 pts on boundary & 2 pts as diameter.

Obs C is unique & 2 pts as diameter.

$O(n^4)$ time brute force

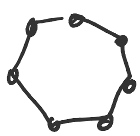
Pf Sketch: By contradiction.



□

Known - $O(n^4)$ by brute force

- compute convex hull ← $O(n \log n)$
  then solve prob for convex polygon? how?



- compute a "Voronoi diagram" ← $O(n \log n)$

- Megiddo / Dyer '83:
  $O(n)$ time! COMPLICATED..

Simpler rand. $O(n)$ alg'm ...

Subproblem (in 1D) Given n distinct numbers $S = \{x_1, \ldots, x_n\}$,
find the min.

Standard "incremental" alg'm:
randomly permute $x_1, \ldots, x_n$

Standard "incremental" algm:

```
0. randomly permute x_1, ..., x_n
1. ans = ∞
2. for i = 1 to n do
3.     if x_i < ans then
4.         ans = x_i          (*)
```

Question: how many switches (*)?

Ans: n in worst-case.

New Question: expected # switches?

Rewrite algm:

```
min(S):        // "backwards" version
0. if S = ∅ return ∞
1. pick x ∈ S randomly
2. ans = min(S - {x})
3. if x < ans then
4.     ans = x              (*)
5. return ans
```

for any fixed S,

$$\Pr[(*) \text{ is done}] = \Pr(x = \min(S))$$
$$= \frac{1}{n}$$

Expected total # switches satisfies

$$T(n) = T(n-1) + \frac{1}{n}$$

$$\Rightarrow T(n) = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}$$

Harmonic number!

$$\Rightarrow T(n) = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot 1$$

Harmonic number!

$$\leq \ln n + 1$$

$$= \boxed{O(\log n)}.$$

Back to 2D . . .

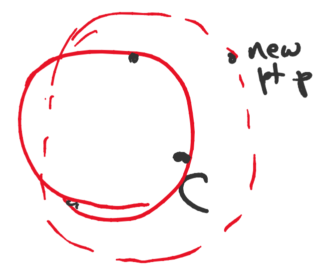
Seidel - Welzl's Las Vegas Alg'm ('91)

idea - incrementally add pts

if p is inside current circle

unchanged

else new circle must have p on boundary



Pf Sketch: by contra. □