2 recursive calls $\frac{N}{2}$ vals $P_2(e^{-\frac{2\pi i k}{N/2}})$, $k=0,1,\dots,\frac{N}{2}-1$

3. for $k = 0, \dots, N-1$,

output $P(e^{-\frac{2\pi i k}{N}}) = P_1(e^{-\frac{4\pi i k}{N}})\, e^{-\frac{2\pi i k}{N}} + P_2(e^{-\frac{4\pi i k}{N}})$

N vals

known from step 2 — known from step 2

note: if $k > \frac{N}{2}$,

$$e^{-\frac{4\pi i k}{N}} = e^{-\frac{4\pi i}{N}(k - N/2)}$$

because $e^{+\frac{4\pi i}{\cancel{N}}\cdot\frac{\cancel{N}}{2}} = e^{2\pi i} = 1$

$$\Rightarrow T(N) = 2T\left(\frac{N}{2}\right) + O(N)$$

$$\Rightarrow T(N) = \boxed{O(N \log N)}$$

Rmk - above alg'm for Problem A is called <u>Fast Fourier Transform</u> (FFT)

Why? it computes $P(e^{-\frac{2\pi i k}{N}}) = \sum_{j=0}^{N-1} a_j e^{-\frac{2\pi k}{N} j}$ $\forall k = 0, \dots, N-1$

$\left(P(x) = \sum_{j=0}^{N-1} a_j x^j\right)$

(Similar to Fourier transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i x t}\, dt$$
)

Alg'm for Problem B:

Alg'm for Problem B:

approach 1: Similar, $\boxed{O(N \log N)}$ time

approach 2:

inverse-FFT "equiv." to FFT

(inverse Fourier transform

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(x)\, e^{2\pi i t x}\, dx \quad)$$

$$\widehat{f \circ g} = \hat{f} \cdot \hat{g}$$

convol.

$\Rightarrow$ polynomial mult / convolution in $\boxed{O(n \log n)}$ time

Rmk - precision issues ...

$(e^{ix} = \cos x + i \sin x)$

Matrix Multiplication

Given $n \times n$ matrices A, B, compute $C = AB$

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 36 & 42 & x \\ x & x & x \\ x & x & x \end{pmatrix}$$

Obvious alg'm (by def'n):

for $i = 1$ to n
  for $j = 1$ to n
    $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$\boxed{O(n^3)}$ time

better?

(lots of appl'ns: inverse / det / $Ax = b$ . . .)

Strassen's Alg'm ('69)

divide $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ $\quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$

where each A_{ij}, B_{ij} is $\frac{n}{2} \times \frac{n}{2}$

1st idea. $AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$

$\Rightarrow \quad T(n) = 8\,T\left(\frac{n}{2}\right) + O(n^2)$

$\xRightarrow{\text{master thm}} \quad O(n^{\log_2 8}) = O(n^3)$

not better ☹

more clever idea -

$$C_1 = A_{11}(B_{11} - B_{21}) \qquad C_2 = (A_{11} + A_{12}) B_{21}$$

$$C_3 = A_{22}(B_{22} - B_{12}) \qquad C_4 = (A_{21} + A_{22}) \cancel{B_{22}}\ B_{12}$$

(typo fixed)

$$C_5 = (A_{11} + A_{22})(B_{21} + B_{12})$$

$$C_6 = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$C_7 = (A_{21} - A_{11})(B_{11} + B_{12})$$

$$AB = \left(\begin{array}{c|c} C_1 + C_2 & C_5 + C_6 + C_3 - C_2 \\ \hline C_5 + C_7 + C_1 - C_4 & C_3 + C_4 \end{array}\right)$$

$$\Rightarrow \quad T(n) = 7\,T\left(\frac{n}{2}\right) + O(n^2)$$

$$\Rightarrow \quad O(n^{\log_2 7}) \le \boxed{O(n^{2.81})}$$

better?

3-way D&C: $T(n) = 23\,T\left(\frac{n}{3}\right) + O(n^2)$

$\Rightarrow \quad O(n^{2.85})$ worse!

⋮

Pan '78: $T(n) = \underline{143640}\ T\left(\frac{n}{70}\right) + O(n^2)$ k

$\frac{k^3 - 4k}{3} + 6k^2$

$\Rightarrow \quad O(n^{2.795})$

'79: $O(n^{2.781})$

'80: $O(n^{2.780})$ ←

'81: $O(n^{2.522})$

Strassen '86: $O(n^{2.479})$

'90: $O(n^{2.376})$

Strassen '86: $O(n^{2.479})$

Coppersmith - Winograd '90: $O(n^{2.376})$

Stothers '10 $O(n^{2.373})$

Vassilevska W. '12:

$O(n^{2.37287})$

Le Gall '14:

Alman, Vassilevska W. '21: $O(n^{2.37286})$