CS 473, Spring 2023
Homework 8 (due April 12 Wed 9pm)

Instructions: As in Homework 1.

Problem 8.1: Consider the following polyhedra P defined by the following set of inequalities:

$$\begin{aligned} x + 2y &\leq 12 \\ -x + 3y &\leq 9 \\ 2x - 3y &\leq 8 \\ x, y &\geq 0 \end{aligned}$$

(a) Draw P on $\mathbb{R}^2$.

(b) Identify all the vertices of P.

(c) Solve the linear program $\pi = \max\{2x + y : (x, y) \in P\}$. Give a proof of correctness, using the fundamental theorem of linear programming.

(d) Write the dual of π.

Problem 8.2: We are given n tasks to perform. Task i requires p_i units of power consumption for a duration of h_i hours. At any moment in time, we can perform at most 3 different tasks, and at any moment in time, the total power consumption must be at most P. A task may be preempted (possibly multiple times) at no extra cost. The problem is to devise a schedule to perform all n tasks with the minimum total number of hours.

For example for $n = 5$ with $p_1 = 10$, $h_1 = 8.5$, $p_2 = 20$, $h_2 = 9$, $p_3 = 60$, $h_3 = 4$, $p_4 = 80$, $h_4 = 3.5$, $p_5 = 90$, $h_5 = 2$, and $P = 100$, one feasible solution is to do tasks 1 and 5 for 2 hours, then tasks 2 and 4 for 3.5 hours, then tasks 1, 2, and 3 for 4 hours, then tasks 1 and 2 for 1.5 hours, and finally task 1 for 1 hour; the total number of hours is 12. (I did not check if this is optimal. Also, for this small example, the constraint that we can do at most 3 tasks at any time is not important; but it could make a difference on larger instances.)

Describe how to solve this problem using linear programming. The number of variables and constraints should be polynomial in n.

(Hint: aim for $O(n^3)$ variables.)

Problem 8.3: We need to divide m tasks among n agents. The tasks are *divisible*, in the sense that it could be done by multiple agents (for example, we could divide up task 1 and let agent 1 do 0.1 of task 1, agent 2 do 0.7 of task 1, and agent 3 do 0.2 of task 1).

For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, agent i incurs a *disutility* of $d_{ij} > 0$ units for doing all of task j (so, if agent i does 0.1 of task j, then task j would contribute $0.1d_{ij}$ units to the disutility of agent i). Our goal is to divide the tasks in a *fair* way, i.e., we wish to divide the tasks such that the disutility of the least happy agent (the one with highest disutility) is minimized.

(a) Write an LP for the above problem. You should have a total of $m + n$ constraints and your LP should be in the standard form ($\min c^T x$ subject to $Ax \geq b,\ x \geq 0$, and the number of rows of A should be $m + n$).

(b) Construct the dual of the LP.