Instructions: As in Homework 1.

Problem 10.1: We are given a set of \( n \) points \( P \) in \( \mathbb{R}^2 \), and a set of \( m \) triangles \( T \). We say that a point \( p \in P \) stabs a triangle \( \Delta \in T \) if and only if \( p \in \Delta \). Each point \( p \) stabs at most \( \alpha \) triangles. We want to find \( M \subset P \) of smallest size that stabs all triangles in \( T \). Let \( \text{OPT} = |M| \).

Consider the iterative algorithm, which in each iteration, adds the point that stabs the maximum number of unstabbed triangles yet. We analyze the approximation quality of this algorithm. Let \( s_i \) be the number of triangles stabbed by the point in the \( i \)th iteration.

(a) Prove that \( s_1 \geq s_2 \geq \cdots \geq s_k \).

(b) We define an epoch as a sequence of iterations from \( u \) to \( v \) \((v \geq u)\), such that \( s_v \geq s_u / 2 \).

Find an upper bound on the length of the epoch \( v - u + 1 \) as a function of \( \text{OPT} \).

(c) Find an upper bound on the number of epochs as a function of \( \alpha \).

(d) Use the results in (b) and (c) to bound the approximation ratio of the algorithm.

Problem 10.2: We want to divide a set of indivisible items \( S = \{g_1, g_2, \ldots, g_m\} \) among \( n \) agents. Each agent \( i \) has a utility of \( u_{ij} \) for good \( g_j \). We know that \( \sum_{j \in [m]} u_{ij} = n \) for all \( i \in [n] \), and \( 0 \leq u_{ij} \leq 1/2 \) for all \( i \in [n], j \in [m] \). (The notation \([n] \) stands for \( \{1, 2, \ldots, n\} \).)

Given an allocation \( X = \langle X_1, X_2, \ldots, X_n \rangle \)\(^2\) we define the max-min share (MMS) of an agent \( i \) to be the maximum over all partitions, the least valued bundle, according to the utility function of this agent, i.e.,

\[
\text{MMS}_i = \max_{X = \langle X_1, X_2, \ldots, X_n \rangle} \min_{j \in [n]} u_i(X_j)
\]

where \( u_i(X_j) = \sum_{g \in X_j} u_{ig} \). Intuitively, this is what agent \( i \) considers her fair share. Our goal is to find an allocation that gives every agent a good approximation of their MMS. An allocation \( X \) is said to be \( \alpha \)-approximate MMS if \( u_i(X_i) = \sum_{g \in X_i} u_{ig} \geq \alpha \text{MMS}_i \). In this exercise, we find a 1/2-approximate MMS allocation.

(a) Show that \( \text{MMS}_i \leq 1 \) for all agents \( i \).

(b) Consider the following iterative process: Add items iteratively into an empty bag \( B \) until one agent \( i \) values the bag at 1/2, i.e., \( u_i(B) \geq 1/2 \). Assign items in this bag to this agent \( i \), i.e., set \( X_i = B \), and continue the same process among the rest of the agents (now there is one less agent). Argue that every agent gets assigned a set of items with a total utility of 1/2.

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\(^1\)More precisely, the number of triangles that are stabbed by the point in the \( i \)th iteration and not stabbed by the points from earlier iterations.

\(^2\)Each \( X_i \subseteq S \) is the bundle allocated to agent \( i \).
(c) Use (a) and (b) to argue that every agent gets at least 1/2 their MMS.

**Problem 10.3:** Given a directed graph $G = (V, E)$, we want to find a subgraph $A$ that is acyclic (i.e., a dag), maximizing the number of edges in $A$.

(a) Show that the following deterministic algorithm yields a feasible solution with approximation ratio at least $1/2$:

1. fix an arbitrary (not random) order of the vertices $v_1, \ldots, v_n$
2. for $i = 2$ to $n$ do {
3. \hspace{1em} $IN_i =$ all edges from $\{v_1, \ldots, v_{i-1}\}$ to $v_i$
4. \hspace{1em} $OUT_i =$ all edges from $v_i$ to $\{v_1, \ldots, v_{i-1}\}$
5. \hspace{1em} if $|IN_i| \geq |OUT_i|$ then insert $IN_i$ to $A$ else insert $OUT_i$ to $A$
}

(b) Show that the following randomized algorithm yields a feasible solution and analyze its expected approximation ratio:

1. take a random order of the vertices $v_1, \ldots, v_n$
2. for each edge $(v_i, v_j) \in E$ do
3. \hspace{1em} if $i < j$ then insert $(v_i, v_j)$ to $A$