CS 473, Spring 2023
Homework 10 (due Apr 26 Wed 9pm)

Instructions: As in Homework 1.

Problem 10.1: We are given a set of n points P in $\mathbb{R}^2$, and a set of m triangles T. We say that a point $p \in P$ *stabs* a triangle $\Delta \in T$ if and only if $p \in \Delta$. Each point p stabs at most α triangles. We want to find $M \subset P$ of smallest size that stabs all triangles in T. Let $OPT = |M|$.

Consider the iterative algorithm, which in each iteration, adds the point that stabs the maximum number of unstabbed triangles yet. We analyze the approximation quality of this algorithm. Let s_i be the number of triangles stabbed by the point in the i^{th} iteration.[1]

(a) Prove that $s_1 \geq s_2 \geq \cdots \geq s_k$.

(b) We define an *epoch* as a sequence of iterations from u to v ($v \geq u$), such that $s_v \geq s_u/2$. Find an upper bound on the length of the epoch $v - u + 1$ as a function of OPT.

(c) Find an upper bound on the number of epochs as a function of α.

(d) Use the results in (b) and (c) to bound the approximation ratio of the algorithm.

Problem 10.2: We want to divide a set of *indivisible* items $S = \{g_1, g_2, \dots, g_m\}$ among n agents. Each agent i has a utility of u_{ij} for good g_j. We know that $\sum_{j\in[m]} u_{ij} = n$ for all $i \in [n]$, and $0 \leq u_{ij} \leq 1/2$ for all $i \in [n], j \in [m]$. (The notation $[n]$ stands for $\{1, 2, \dots, n\}$.)

Given an allocation $X = \langle X_1, X_2, \dots, X_n\rangle$[2], we define the *max-min share* (MMS) of an agent i to be the maximum over all partitions, the least valued bundle, according to the utility function of this agent, i.e.,

$$MMS_i = \max_{X=\langle X_1,X_2,\dots,X_n\rangle} \min_{j\in[n]} u_i(X_j)$$

where $u_i(X_j) = \sum_{g\in X_j} u_{ig}$. Intuitively, this is what agent i considers her *fair share*. Our goal is to find an allocation that gives every agent a good approximation of their MMS. An allocation X is said to be *α-approximate MMS* if $u_i(X_i) = \sum_{g\in X_i} u_{ig} \geq \alpha MMS_i$. In this exercise, we find a 1/2-approximate MMS allocation.

(a) Show that $MMS_i \leq 1$ for all agents i.

(b) Consider the following iterative process: Add items iteratively into an empty bag B until one agent i values the bag at 1/2, i.e., $u_i(B) \geq 1/2$. Assign items in this bag to this agent i, i.e., set $X_i = B$, and continue the same process among the rest of the agents (now there is one less agent). Argue that every agent gets assigned a set of items with a total utility of 1/2.

[1] More precisely, the number of triangles that are stabbed by the point in the i^{th} iteration and not stabbed by the points from earlier iterations.

[2] Each $X_i \subseteq S$ is the bundle allocated to agent i.

(c) Use (a) and (b) to argue that every agent gets at least 1/2 their MMS.

Problem 10.3: Given a *directed* graph $G = (V, E)$, we want to find a subgraph A that is acyclic (i.e., a dag), maximizing the number of edges in A.

(a) Show that the following deterministic algorithm yields a feasible solution with approximation ratio at least 1/2:

```
fix an arbitrary (not random) order of the vertices v_1, ..., v_n
for i = 2 to n do {
    IN_i = all edges from {v_1, ..., v_{i-1}} to v_i
    OUT_i = all edges from v_i to {v_1, ..., v_{i-1}}
    if |IN_i| ≥ |OUT_i| then insert IN_i to A else insert OUT_i to A
}
```

(b) Show that the following randomized algorithm yields a feasible solution and analyze its expected approximation ratio:

```
take a random order of the vertices v_1, ..., v_n
for each edge (v_i, v_j) ∈ E do
    if i < j then insert (v_i, v_j) to A
```