cs473 Algorithms	Out: *Fri., 2022-04-15 17:00*

Problem Set #9

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Due: *Fri., 2022-04-22 17:00*

All problems are of equal value.

1. Consider the polyhedra defined by

$$P = \{(x,y) \mid \quad x + 2y \le 12, \\ -x + 3y \le 9, \\ 2x - 3y \le 8, \\ x, y \ge 0 \}$$

   (a) Prove that P is a *polytope*, in that P is contained in a finite volume bounding box $P \subseteq [-B, B]^2$ for some sufficiently large $B > 0$.

   (b) Give a list of all vertices of the polytope P. Prove that the list is correct.

   (c) Draw P as a subset of $\mathbb{R}^2$.

   (d) Solve the linear program $\Pi = \max_{(x,y)\in P} 2x + y$. Give a proof of correctness, assuming facts stated in lecture.

   (e) Derive the canonical form dual $\amalg$ to Π.

2. Consider (again) the polyhedra P defined above. **Showing your work**, use Fourier-Motzkin elimination to compute the following quantities.

   (a) Compute the projection $\pi_x(P)$ of P onto the x-axis.

   (b) Compute the projection $\pi_y(P)$ of P onto the y-axis.

   (c) Solve the linear program $\Pi = \max_{(x,y)\in P} 2x + y$.

3. Alice and Bob play the following game. Alice chooses a number $i \in \{1, \ldots, n\}$, while Bob *simultaneously* chooses a number $j \in \{1, \ldots, m\}$. The game is governed by the payoff-matrix $A \in \mathbb{Z}^{n\times m}$, so that Alice wins $A_{i,j}$ and Bob wins $-A_{i,j}$ (hence this is a *zero sum* game).

   (a) Suppose that Alice has a probabilistic strategy, where $i \in [n]$ is played with probability p_i. Thus, $p_i \ge 0$ and $\sum_{i\in[n]} p_i = 1$. Suppose further that Bob knows this strategy. Write a linear program to determine the probabilities $(q_j)_{1\le j\le m}$ that Bob should play to maximize his expected payoff.

   (b) Write the dual of the above LP.

   (c) Suppose now that while Bob is playing against Alice, Bob is also playing the same game against Charlie. However, the payoff matrix against Charlie is now $C \in \mathbb{Z}^{n\times m}$, so that if Charlie plays $k \in [n]$ and Bob plays $j \in [m]$ that Charlie wins $C_{k,j}$ and Bob wins $-C_{k,j}$. Suppose that Bob is constrained to play the *same* strategy in both games. Supposing that Alice plays the probabilistic strategy $(p_i)_{i\in[n]}$, Charlie plays the probabilistic strategy $(r_k)_{k\in[n]}$, write an LP to determine the best probabilistic strategy for Bob that maximizes the minimum of the expected payoffs in the two games.