CS473 Algorithms: Lecture 9 (2022-02-15)

logistics:
- pset 3 due Fri
- async this week
  - ↳ my office hrs cancelled

last lecture:
- flows
  - review
  - cuts
  - max-flow ≤ min-cut
  - ____ = ____ iff G^f no $s \rightsquigarrow t$ path
    - ⟺ FF correct
  - ____ = ____, for [integral] capacities

today: flows

Q: compute max (s,t)-flow?

idea: repeatedly augment flow, via augmenting path in residual graph

algo (Ford Fulkerson)

(1) $f_e \leftarrow 0$, $e \in E$

(2) initialize G^f

(3) while exists augmenting p in G^f

  (a) $f \leftarrow f + p$

  (b) $G^f \leftarrow G^{f+p}$

(4) return f

prop: [any] flow f, $|f| \le \sum_e c_e = F$

prop: Ford Fulkerson takes $O(F)$ iterations

$O(mF)$ time

def: (s,t)-cut C in G is partition $V = (S,T)$, $s \in S$, $t \in T$

the capacity of C is $|C| = \sum_{e: u \to v,\ u \in S,\ v \in T} c_e$

def: f flow in G. $S \subseteq V$

the flow through S is

$$f(S) = \underbrace{f^{out}(S)}_{\sum_{e: u \to v,\ u \in S,\ v \notin S} f_e} - \underbrace{f^{in}(S)}_{\sum_{e: u \leftarrow v,\ u \in S,\ v \notin S} f_e}$$

prop: f (s,t)-flow, $C = (S,T)$ (s,t)-cut. then

$$|f| := f(s) = \sum_{v \in S} f(v) = f(S) = \underbrace{f^{out}(S)}_{\le |C|} - \underbrace{f^{in}(S)}_{\ge 0} \le |C|$$

$\implies \max_f |f| \le \min_C |C|$

prop: f flow in G. $S = \{u : s \rightsquigarrow u \text{ in } G^f\}$. equiv

(1) G^f has no $s \rightsquigarrow t$ path

(2) $C = (S, \underbrace{V \setminus S}_{T})$ is (s,t)-cut, $|C| = |f|$

(3) f max flow

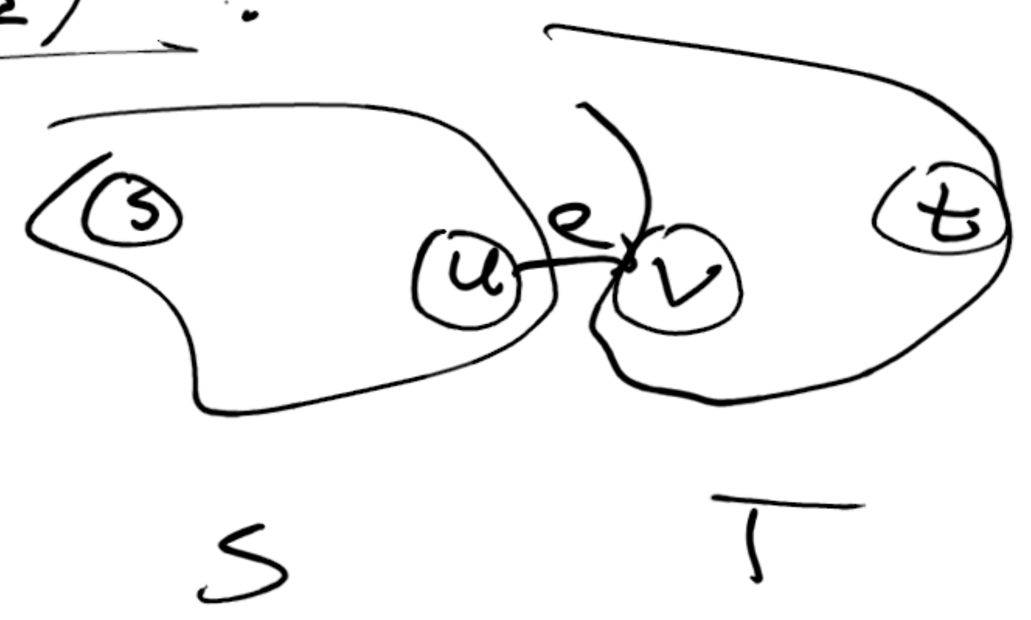
sketch: $\neg(1) \Rightarrow \neg(3)$: $|f+p| = |f| + |p| > |f|$

$(2) \Rightarrow (3)$: $\max |f| \le \min |C|$

$\Rightarrow$ if equality then both optimal

$(1) \Rightarrow (2)$:

G:



G^f:

$\Rightarrow t \in T$

$\Rightarrow e \notin E^f$

$\Rightarrow f_e = c_e$

G:

G^f:

$\Rightarrow e \notin E^f$

$\Rightarrow f_e = 0$

$\ldots \Rightarrow |f| = |C|$ □

cor: FF terminates w/ max flow in $O(mF)$ steps

thm [max-flow min-cut theorem]

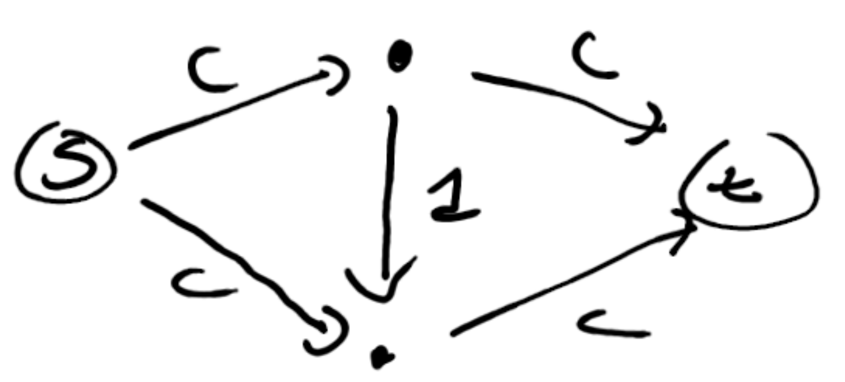
G capacitated graph w/ integral capacity

then $\max\limits_{f\ (s,t)\text{-flow}} |f| = \min\limits_{C\ (s,t)\text{-cut}} |C|$

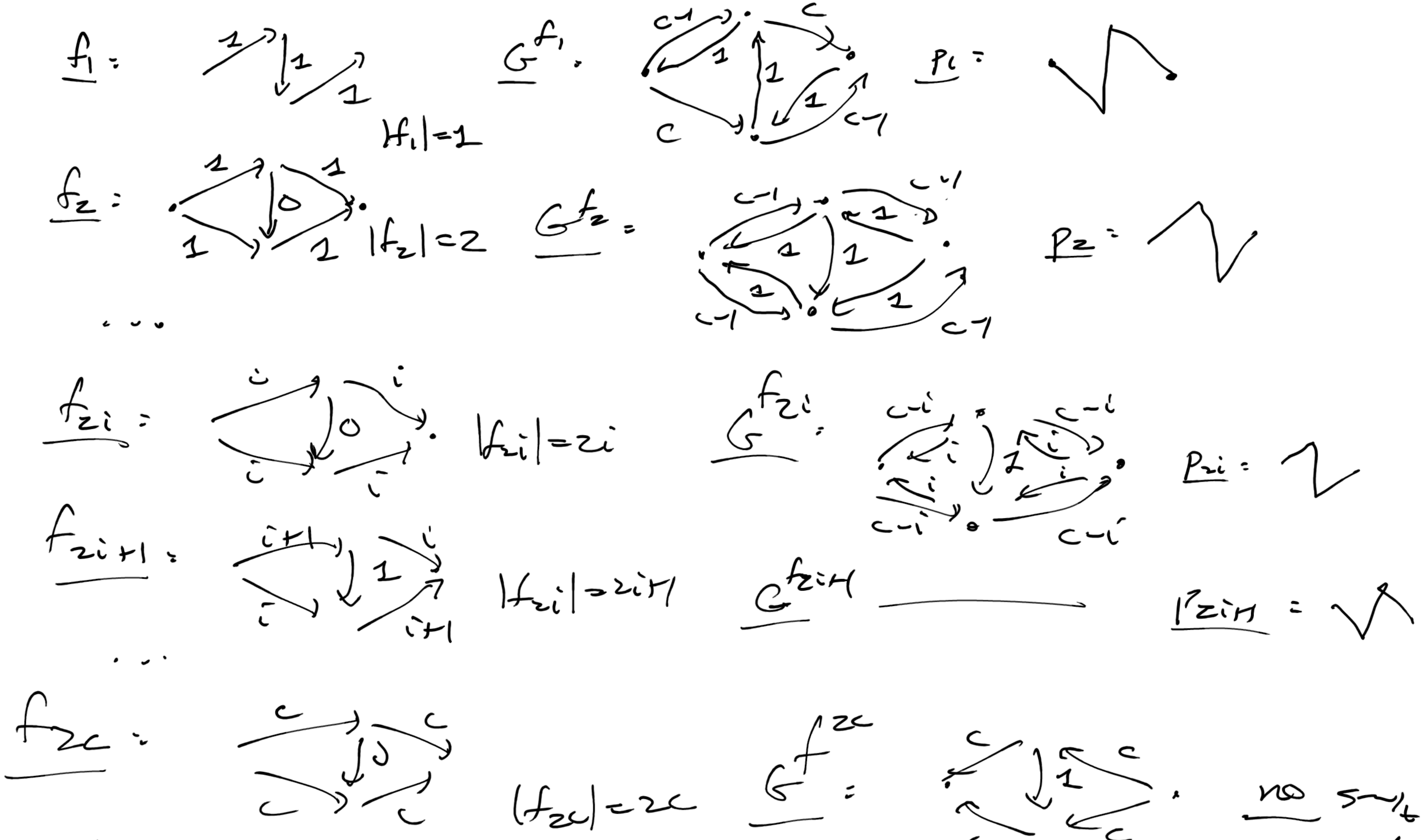
Q: are we done?

Claim: exists graphs where FF takes $\Omega(F)$ iterations

pf: G:



$F = \Theta(c)$



no s-t path

$\Rightarrow$ 2c iterations

rmk: **but** 2 iterations suffice

$\Rightarrow$ some augmenting path choices are **bad** def: for a problem on sequences of n integers $a_1, \ldots, a_n$, an algorithm runs in **polynomial time** if it runs in $\text{poly}(\sum \log a_i)$ time

pseudo ———————— $(\sum a_i)$ —

rmk: - sometimes pseudopoly algo are interesting

↳ knapsack DP algo was pseudopoly

- often pseudopoly time are not efficient

1000 vs 1000

$\underbrace{1000}_{\text{SI binary}}$ 4 vs $\underbrace{1000}_{\text{SI unary}}$ 1000

cor: maxflow has [pseudo] poly time algo

Q: poly time algo?

prop: given f flow in G w/ $|f| \geq |f^*| - B$

$\Rightarrow$ can find max-flow in $O(mB)$ steps

sketch: run FF starting w/ f

$\Rightarrow$ only need $\leq B$ iterations

Cor: can find max flow in $O(m|f^*|)$ time

rmk: many natural problems have $|f^*| \in \text{poly}(n,m)$

Q: poly time in general?

idea: find (good) augmenting paths

↳ large value

def: f flow in G, residual graph G^f is ...

the **Δ-bottleneck residual graph** is $G^{f,\Delta}$, given by

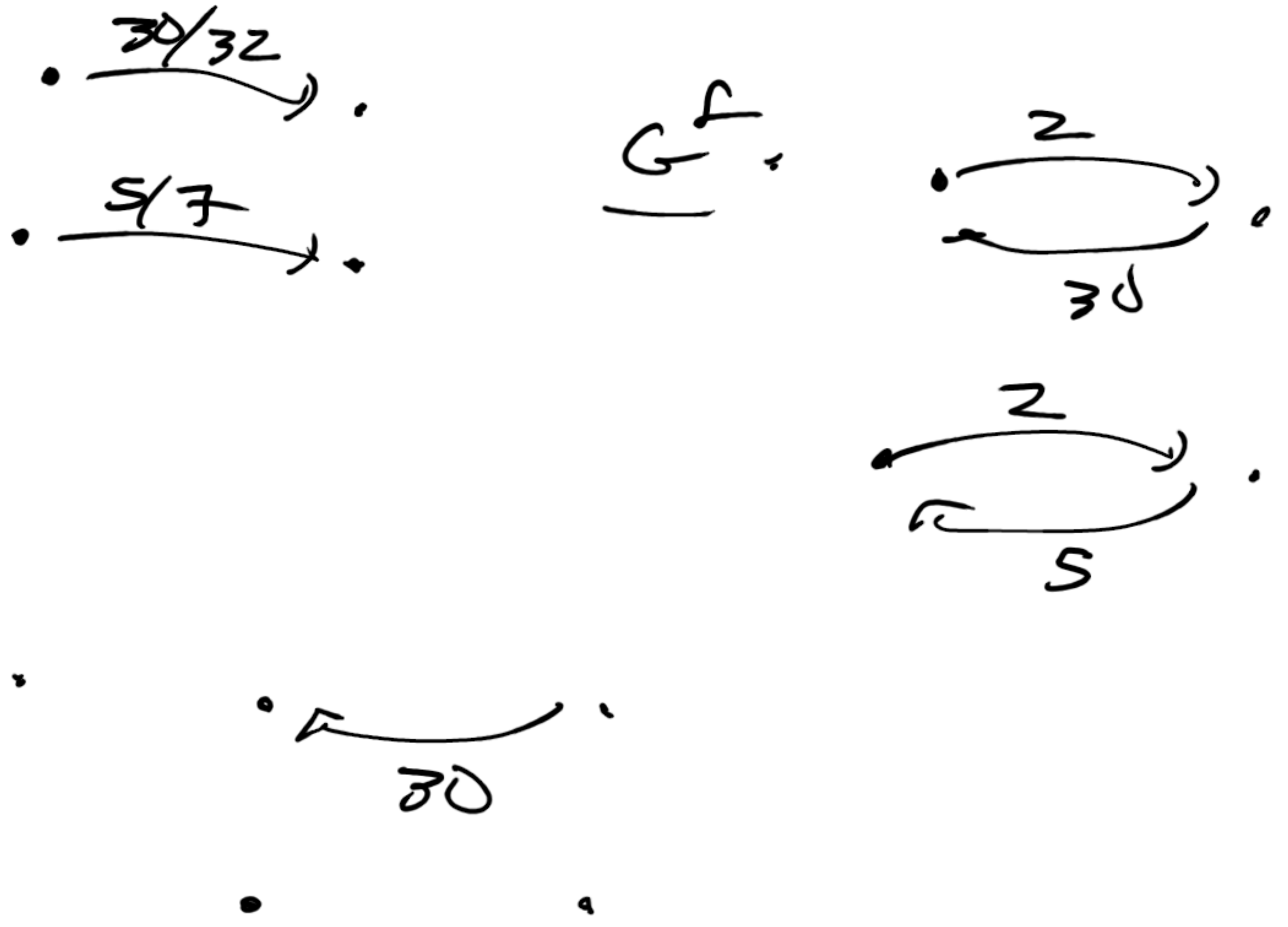
$V^{f,\Delta} = V^f$

$c^{f,\Delta} = c^f$

$E^{f,\Delta} = \{e \in E^f : (c^f)_e \geq \Delta\}$

rmk: $G^{f,1} = G^f$

eg: G: , G^f: , $G^{f,15}$:



prop: $s \rightsquigarrow t$ path in $G^{f,\Delta}$

(1) are $s \rightsquigarrow t$ paths in G^f

(2) are augmenting paths P w/ $|P| \geq \Delta$

Q: what is Δ?

idea (scaling): start w/ large Δ

make Δ smaller over time

also (scaling-FF):

(1) initialize f, G^f

(2) $F = \sum_e c_e$

(3) $\Delta = 2^{\lfloor \log F \rfloor}$

(4) init $G^{f,\Delta}$

(5) while $\Delta \geq 1$

(a) while $s \rightsquigarrow t$ path in $G^{f,\Delta}$

(i) $f \leftarrow f + p$

(ii) update G^f, $G^{f,\Delta}$

(b) $\Delta \leftarrow \Delta / 2$

(6) return f

prop: scaling FF terminates w/ max flow

sketch: scaling-FF instantiates FF w/ rule for choosing "good" augmenting paths

eventually $\Delta = 1 \Rightarrow$ all augmenting paths considered

$\Rightarrow$ any run of scaling FF is a run of FF $\leftarrow$ FF terminates w/ max flow

Q: complexity?

<u>idea</u>: use max flow = min cut

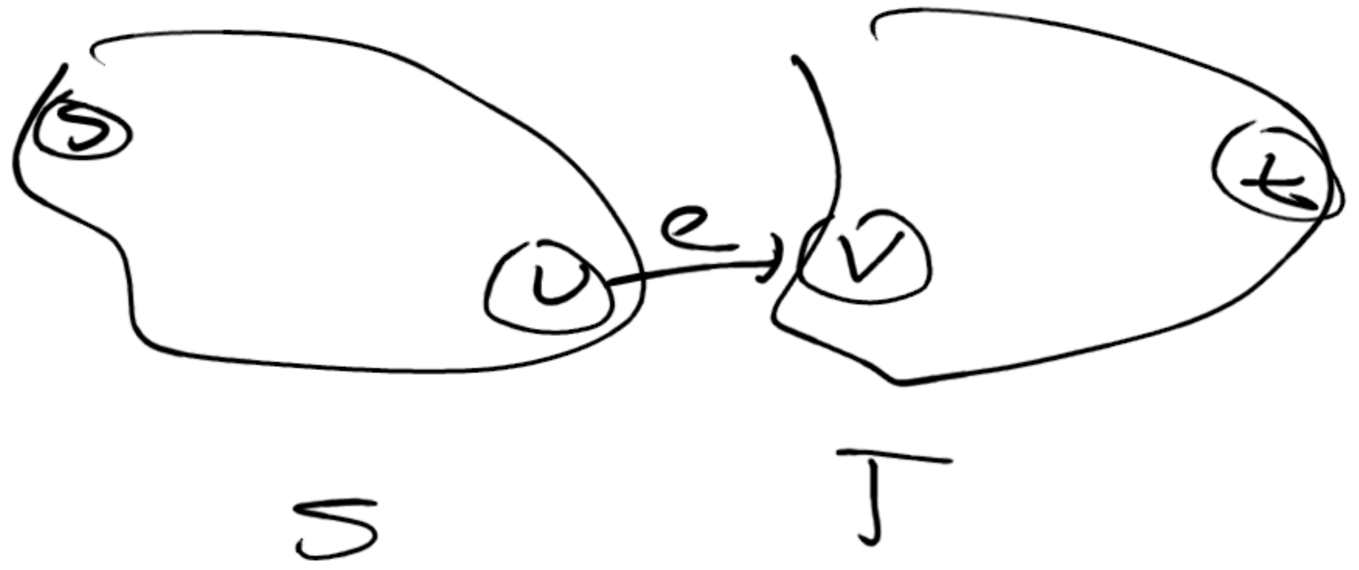
<u>prop</u>: f flow in G, $G^{f,\Delta}$ has no $s \rightsquigarrow t$ path

$$\Rightarrow |f| \geq |f^*| - m(\Delta - 1)$$

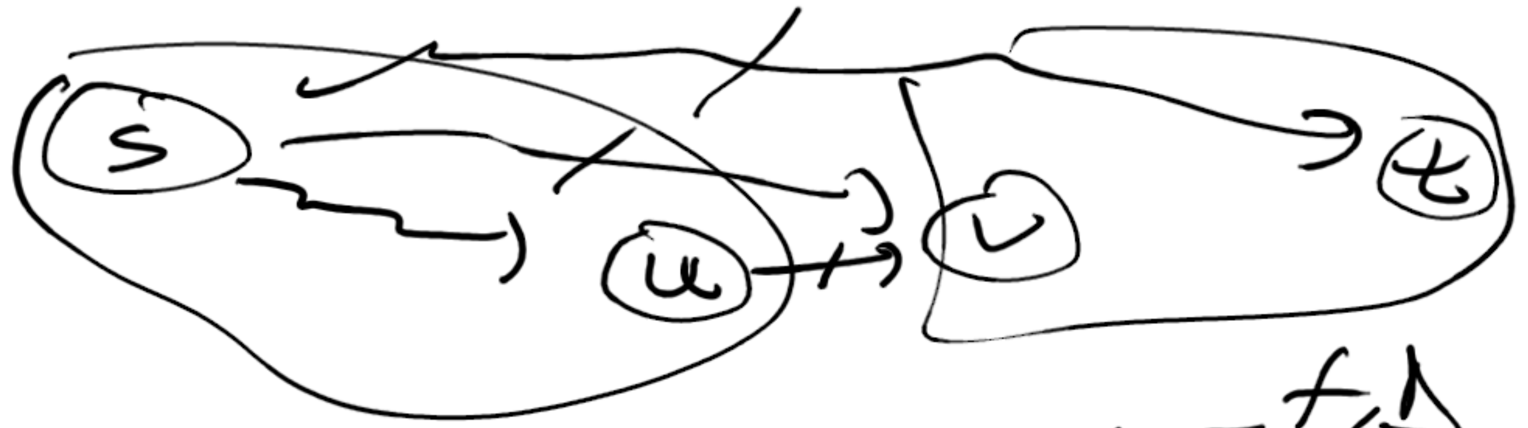
<u>sketch</u>: $S = \{ u : s \rightsquigarrow u \text{ in } G^{f,\Delta} \}$

<u>clm</u>: $(S, T = V \setminus S)$ is (s,t)-cut

<u>G</u>:



<u>$G^{f,\Delta}$</u> =



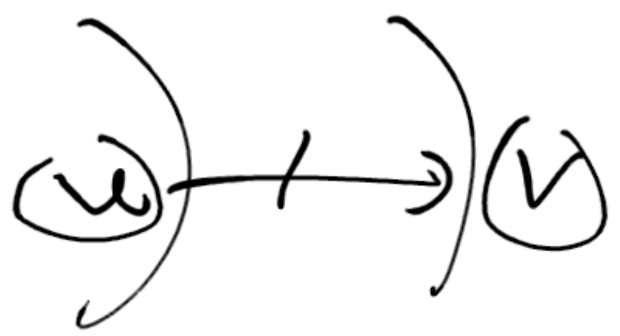
$\Rightarrow e \notin E^{f,\Delta}$

$\equiv \underbrace{(c^f)_e}_{c_e - f_e} < \Delta$

$\equiv f_e \geq c_e - (\Delta - 1)$

<u>G</u>:

<u>$G^{f,\Delta}$</u> :



$\Rightarrow -e \notin E^{f,\Delta}$

$\equiv \underbrace{(c^f)_e}_{f_e} < \Delta \Rightarrow f_e \leq \Delta - 1$

$$\Rightarrow |f| = f(s) = f(S) = f^{out}(S) - f^{in}(S)$$

$$= \sum_{\substack{e : u \to v \\ u \in S,\ v \in T}} \underbrace{f_e}_{\geq c_e - (\Delta-1)} - \sum_{\substack{e : u \leftarrow v \\ u \in S,\ v \in T}} \underbrace{f_e}_{\leq \Delta - 1}$$

$$\geq \sum_{\substack{e : u \to v \\ u \in S,\ v \in T}} c_e - \underbrace{\sum_{e \in E} (\Delta - 1)}_{\leq m(\Delta-1)} \geq \underbrace{|C|}_{\geq |f^*|} - m(\Delta - 1)$$

☒

cor. f flow in G, no s→t path in $G^{f,2\Delta}$ ⇒ any sequence of ($\geq \Delta$)-value augmentations to f has $\leq 2m$ paths

pf. no s→t path in $G^{f,2\Delta}$ ⇒ $|f| \geq |f^*| - 2m\Delta$

each ($\geq \Delta$)-value augmentation adds $\geq \Delta$ to the value

⇒ k augmentations yield at least $|f| + k\Delta \leq |f^*|$

$\leq |f| + 2m\Delta$

⇒ $k \leq 2m$

cor. scaling FF runs in $O(m^2 \lg F)$ time

pf: algo.

$\Delta = 2^{\lfloor \lg F \rfloor}$

while $\Delta \geq 1$ ← $O(\lg F)$ iteration

while s→t path in $G^{f,\Delta}$

augment ← $O(m)$ time

$\Delta \leftarrow \Delta/2$

clm: $O(m)$ augmentations per round

pf: $\Delta = 2^{\lfloor \lg F \rfloor}$: $2\Delta > F \geq |f^*|$. Any augment via $G^{f,\Delta}$ adds $\geq \Delta$ in value ⇒ $O(1)$ augmentations here

$\Delta = 2^i$:

at end of last scaling round no s→t path in $G^{f,2\Delta}$

⇒ $\leq 2m$ further augmentations of ($\geq \Delta$) value

this scaling round ☐

rmk: runs as before max flow obs

today: flows
- review
- FF is pseudopoly
- [bad] asymmetry poly
- scaling FF

next tues: flows

logistics: - pset 3 due F 12