More on Approximation Algorithms (part 3)

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Recap:

Randomized Appx Alg + LP-based appx-based

SCHEDULING → 2 appx (LP-based gen) ; → 3/2 appx (greedy heur.)

SET-COVER → O(lg n) (greedy) ; → f-approx. (LP based)

VERTEX-COVER → 2 appx (LP based)

KNAPSACK-PROB → $(1+\epsilon)$-appx (DP based)

Today → MAX-SAT → 3/4 random Appx alg

SAT / MAX-SAT

Given: → 1) n variables $x_1, x_2, \ldots, x_n$

2) m clauses $C_1, C_2, \ldots, C_m$

$C_i = (l_{i_1} \vee l_{i_2} \vee \ldots)$

$l_{i_1} = x_i$ or $l_{i_1} = \neg x_i$

3) $w_1, w_2, \ldots$ w_i = weight of clause i

Find: → An assignment [T, F] to

$x_1, x_2, \ldots, x_n$ (that satisfies max number of clause) that maximizes

$$\sum_{i:\ C_i \text{ is satisfied}} w_i$$

SAT $\leq_p$ MAX-SAT (NP-hard)

Unbiased Randomized Algorithm

for $i = 1$ to n.
set $x_i = T$ with probability $\frac{1}{2}$
$x_i = F$ with probability $\frac{1}{2}$ $\Big\}\ \underline{\underline{O(n)}}$

consider $\underline{\underline{C_i}}$

$$\Pr[C_i \text{ is satisfied}] = 1 - \underbrace{\Pr[C_i \text{ is not satisfied}]}$$

$$\Pr[C_i \text{ is not satisfied}] = \left(\frac{1}{2}\right)^{l_i}$$

l_i: # literals in the clause.

$$\Pr[C_i \text{ is satisfied}] = \underline{1 - 2^{-l_i}}$$

$\Pr[C_i \text{ is satisfied}] = \underline{\underline{1 - 2^{-\ell_i}}}$

$\boxed{E[\text{solution}]}$

$= E\left[\sum_{i \in [m]} w_i \, y_i\right]$ $y_i = 1$ if clause C_i is satisfied

$\quad = 0$ if it is not satisfied

$= \sum_{i \in [m]} w_i \, E[y_i]$

$= \sum_{i \in [m]} w_i \cdot \Pr[y_i = 1]$

$= \sum_{i \in [m]} w_i \cdot (1 - 2^{-\ell_i})$

$\ell_i \geq 1$ for all $i \in [m]$. $\underline{\underline{\ell_i = 1}}$ (bad)

$\geq \frac{1}{2} \cdot \sum_{i \in [m]} w_i$

$\geq \underline{\underline{\frac{1}{2} \cdot OPT}}$

$\underline{\underline{3\text{-SAT}}}$ $(1 - 2^{-\ell_i}) = \frac{7}{8} \quad \forall i \in [m]$

3-SAT

$\left(1-2^{-l_i}\right) = \frac{7}{8} \quad \forall i \in (m)$

$l_i = 3 \quad \forall i \in (m)$

same-alg gives a $\frac{7}{8}$-appx for 3SAT

Biased approximation (LP-based approximation)

(I) LP for MAXSAT

$$y_i = \begin{matrix} 1 \\ 0 \end{matrix} \iff \begin{matrix} x_i = T \\ x_i = F \end{matrix} \quad \forall i \in (n]$$

$$z_j = \begin{matrix} 1 \\ 0 \end{matrix} \iff \begin{matrix} c_j \text{ is satisfied} \\ c_j \text{ is unsatisfied} \end{matrix} \quad \forall j \in (m]$$

$$\max \sum_{j \in [m]} w_j \cdot z_j$$

$$\boxed{\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1-y_i) \geq z_j \quad \forall j \in (m]}$$

$$0 \leq z_j \leq 1 \quad \forall j \in (m]$$

$$y_i \in \{0, 1\} \quad \forall i \in [n]$$

c_j → P_j = vars that occur true

→ N_j = vars that occur -vely

every feasible assignment can be converted into a feasible solution for the ILP & the optimum value of the ILP $\geq$ OPT

LP relaxation:

$y_i \in \{0, 1\} \longrightarrow 0 \leq y_i \leq 1$
replace

1) Solve LP relaxation & get y^*, z^*
2) Round y^*, z^* to get a solution for MAX-SAT

UPPER BOUND

$$\left[\sum_{i \in [m]} w_i z_i^* \right. \geq OPT$$

Randomized Rounding

1) set each variable x_i in MAX-SAT

do T with probability $\underline{y_i^*}$, & x_i to F with probability $1-y_i^*$.

$$\underline{z_i'} = \begin{cases} 1 & \text{if } C_i \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_{i \in [m]} w_i z_i'\right]$$

$$= \sum_{i \in [m]} w_i \, E[z_i']$$

$$= \sum_{i \in [m]} w_i \cdot \underline{\Pr[z_i' = 1]}$$

$\alpha \sum w_i \underline{z_i^*}$

$$\Pr[z_i' = 0] \leq \underline{\qquad\qquad}$$

$\underline{\Pr[C_j \text{ is not satisfied}]}$

$\underline{P_j}$ $\quad$ $\underline{N_j}$

$$= \left(\left(\prod (1-x_i^*) \prod x_i^* \right)^{1/\ell_j} \right)^{\ell_j}$$

$$= \left(\left(\underbrace{\prod_{i \in P_j} (1 - y_i^*)}_{|P_j|} \; \underbrace{\prod_{i \in N_j} y_i^*}_{|N_j|} \right)^{1/l_j} \right)^{l_j}$$

$$\leq \left(\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right)^{l_j}$$

$$= \left(\frac{1}{l_j} \left(|P_j| - \sum_{i \in P_j} y_i^* + |N_j| - \sum_{i \in N_j} (1 - y_i^*) \right) \right)^{l_j}$$

$$= \left(\frac{1}{l_j} \left(l_j - \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right) \right)^{l_j}$$

$$= \left(1 - \frac{1}{l_j} \underbrace{\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right)}_{\geq z_j^*} \right)^{l_j}$$

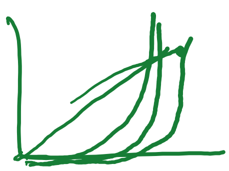
$$\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

Pr[C_j is not satisfied]

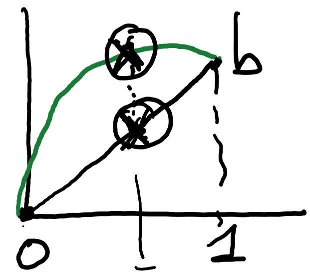
$$\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

$$\Pr[C_j \text{ is satisfied}] \geq 1 - \underbrace{\left(1 - \frac{z_j^*}{l_j}\right)^{l_j}}$$ $\rightarrow$ convex

$0 \leq z_j^* \leq 1$



$$\boxed{b = 1 - \left(1 - \frac{1}{l_j}\right)^{l_j}}$$



$y = bx$

$$\Pr[C_j \text{ is satisfied}] \geq 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

$$\geq b \cdot z_j^*$$

$$= \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) \cdot z_j^*$$

$$E[ALG] \geq \sum_{j \in E(m)} w_j \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^*$$

$\hookrightarrow$ decreasing

in l_j $\uparrow$

$l_i = 1 \quad f(l_i) = 1$

$l_j = 2 \quad f(l_i) = 3/4$

$(1-2^{-l_j}) \rightarrow$ inc- in l_j

$l_j \to \infty \quad f(l_i) \to (1 - 1/e)$

$$E[ALG] \geq (1 - 1/e) \sum_i w_i z_i^{*}$$

$$\geq \underline{(1 - 1/e)} \cdot OPT$$

$\underline{\text{Thm}}$:- Biased Randomized roundn Produces a $(1 - 1/e)$ - appx

$\approx \underline{0.63} \geq 1/2$

<u>Combining both randomized Algorithms</u>

W_1 = val. solution returned by biased r. appx - alg

W_2 = val. of solution returned by the unbiased r. appx - alg

return argmax(W_1, W_2)

$$E[\max(W_1, W_2)]$$

$$\geq E\left[\tfrac{1}{2} W_1 + \tfrac{1}{2} W_2\right]$$

$$= \frac{1}{2}\left\{ E[W_1] + E[W_2] \right\}$$

$$= \frac{1}{2}\left\{ \sum_{j \in [m]} W_j \left(1 - 2^{-l_j}\right) + \sum_{j \in [m]} w_j z_j^* \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) \right\}$$

$$\geq \frac{1}{2}\left\{ \sum_{j \in [m]} w_j z_j^* \left(\underline{\left(1 - 2^{-l_j}\right)} + \underline{\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right)} \right) \right\}$$

Claim: $\underline{\left(1 - 2^{-l_j}\right)} + \underline{1 - \left(1 - \frac{1}{l_j}\right)^{l_j}} \geq 1.5$

$l_j = 1,$ $\quad \frac{1}{2} + 1 = 1.5$

$l_j = 2,$ $\quad \frac{3}{4} + \frac{3}{4} = 1.5$

$l_i = 2$, $\quad \frac{3}{4} + \frac{3}{4} = 1.5$

$l_i \geq 3$, $\quad \frac{7}{8} + 1 - \frac{1}{e} \geq 1.5$

$\approx 0.88 \quad \sim 2\} \; 0.63 \quad 1.5$

$$E[A|G] \geq \frac{1}{2} \cdot \sum_{j \in [m]} w_j z_j^* \times 1.5$$

$$= \frac{3}{4} \sum_{j \in [m]} w_j z_j^*$$

$$= \frac{3}{4} OPT$$

Thm: $\exists$ a randomized $\frac{3}{4}$ appx alg to MAX-SAT

SET-COVER $\quad$ $(O(\lg n))$-appx

LP solution.

Given :- 1) $S_1, S_2, \ldots, S_m \subseteq U$

$|U| = n$

2) each set S_i has weight w_i.

<u>Goal</u>: $\rightarrow$ Find a set of sets C s.t. all elements in U are covered and

$$\sum_{i: S_i \in C} w_i \text{ is minimum}$$

(I) LP

$$x_i = \begin{cases} 1 & \text{if } S_i \in C \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\min \sum_{i \in [m]} w_i x_i}$$

$$\cancel{\forall} \sum_{i: e \in S_i} x_i \geq 1 \quad \forall e \in U$$

$$0 \leq x_i \leq 1$$

1) SOLVE LP to get x^*

2) Round x^* randomly acc to ILP solu

$\Rightarrow$ Rounding a ...
to LP soln

1) select set S_i with prob x_i^*

$y_i = 1$ if S_i is selected
$= 0$ otherwise

$$E[ALG] = E\left[\sum_{i \in [m]} w_i y_i\right]$$

$$= \sum_{i \in [m]} w_i \cdot \Pr[y_i = 1]$$

$$= \sum_{i \in [m]} w_i x_i^* \le \underline{\underline{OPT}}$$