More on NP-completeness

Thursday, April 21, 2022 1:43 PM

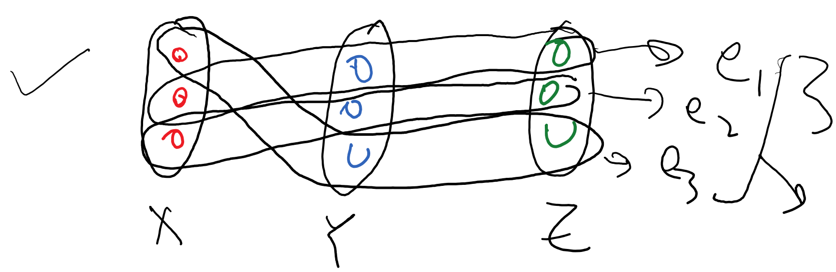
Recap: Saw a lot of NP-complete graph problems.

3DM: **Given:** 1) Partitions $X \cup Y \cup Z$

2) Edges E: each edge $e \in E$, $e = \{i, j, k\}$, $i \in X$, $j \in Y$, $k \in Z$.

3) $|X| = |Y| = |Z|$.

Question: Does there exist a matching M that covers all vertices?



vertex-disjoint edges

M that covers all vertices

Thm: 3DM is NP-complete.

$$3SAT \le_p 3DM$$

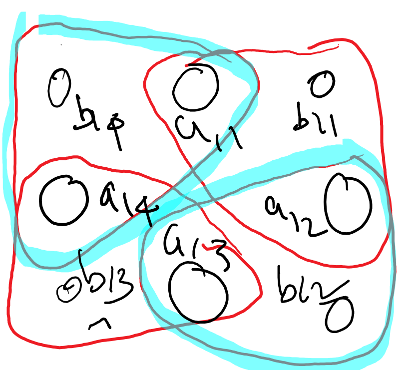
Given an instance I of 3SAT: $\begin{cases} x_1, x_2, \dots x_n & \text{// variables} \\ C_1, C_2, \dots C_m & \text{// clauses} \end{cases}$

variable gadgets:

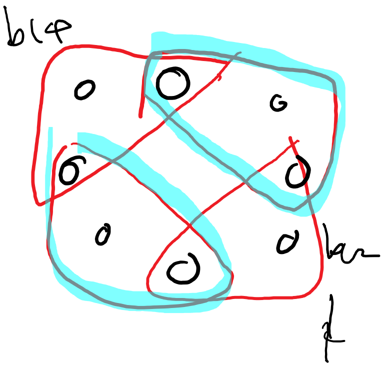
x_i
- → core-vertices = $\{a_{i1}, a_{i2}, \ldots, a_{i2m}\}$
- → tip-vertices = $\{b_{i1}, b_{i2}, \ldots, b_{i2m}\}$
- → triples $= \bigcup_{j=1}^{2m} \{a_{ij}, a_{ij+1}, b_{ij}\}$

(modulo 2m)

$m = 2$: x_1 ?

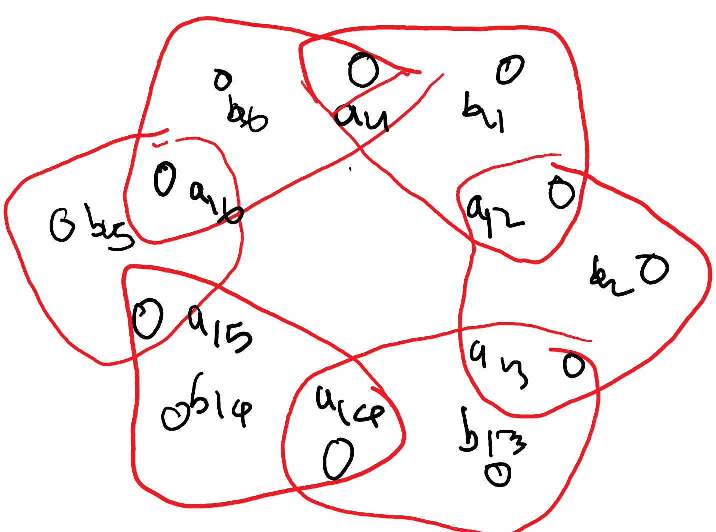


↳ "odd matching in variable gadget"



"even matching in the variable gadget"

$m = 3$



CLAUSE-GADGETS

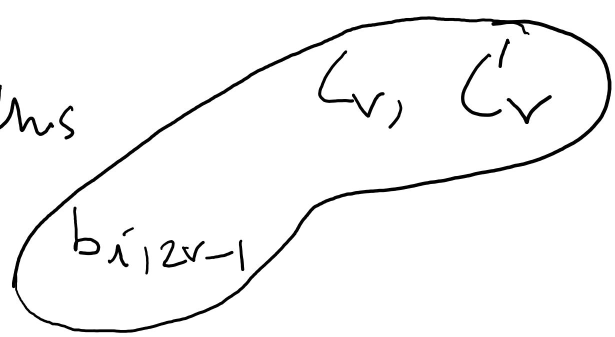
$$C_r = (l_i \vee l_j \vee l_k)$$

$$\{x_i, \neg x_i\} \quad \{x_j, \neg x_j\} \quad \{x_k, \neg x_k\}$$

Create two new vertices C_r, C_r'.
& connect them with variable gadgets appropriately

$l_i = x_i$

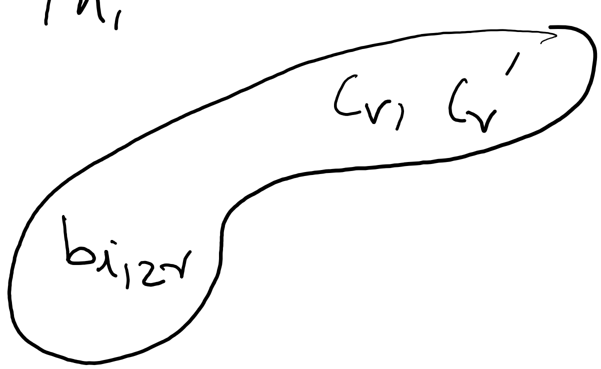
add this edge



$l_j = x_j$ or $\neg x_j$

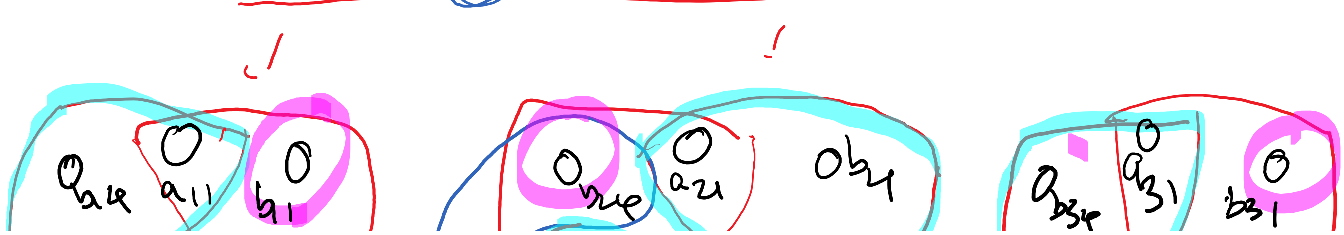
$l_k = x_k$ or $\neg x_k$

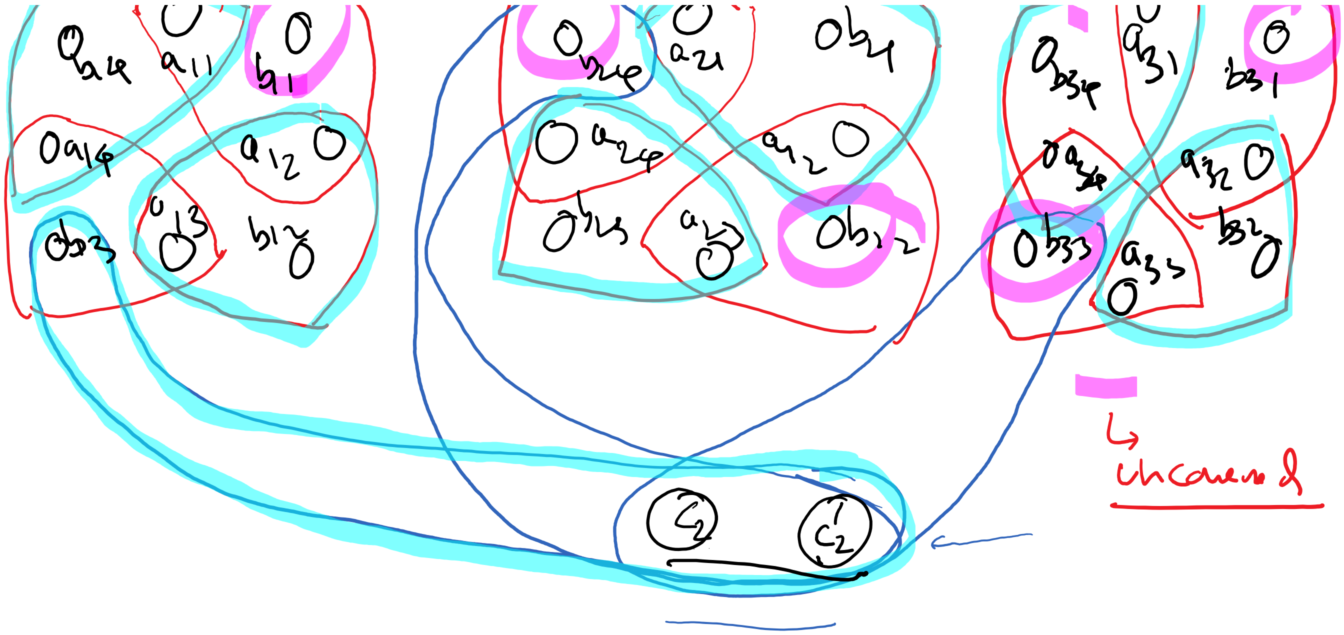
: $l_i = \neg x_i$



$x_1 = T$, $x_2 = F$, $x_3 = T$

$C_2 = (x_1 \vee \neg x_2 \vee x_3)$





Claim: $\exists$ satisfying assignment for I
$\iff$ $\exists$ matching M in I' that covers all vertices

$\Rightarrow$ $x_i = T$ then match var gadget of x_i oddly
$x_i = F$ then match v - x_i evenly

For every clause C_r, we can match the C_r, C_r' with an appropriate tip of some variable.

$2m$ tip vertices for each variable gadget
↳ m tip vertices would be covered

by appr. matching in variable ga

mn tip vertices that remain.

m tip-vertices will be covered by edges from clause-gadgets

$\Rightarrow nm - m = (n-1)m$ tip-vertices will not be covered

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CLEANUP-GADGETS

$(n-1)m$ dummy vertices $\left((G_i, (G'_i) \right)_{i=1}^{(m-1)n}$

Add an edge

$\left((G_i, (G'_i, b \right)$ $i = 1$ to $(n-1)m$
& b is any tip-vertex

"$\Leftarrow$" $\exists$ matching in I' that covers all vertices

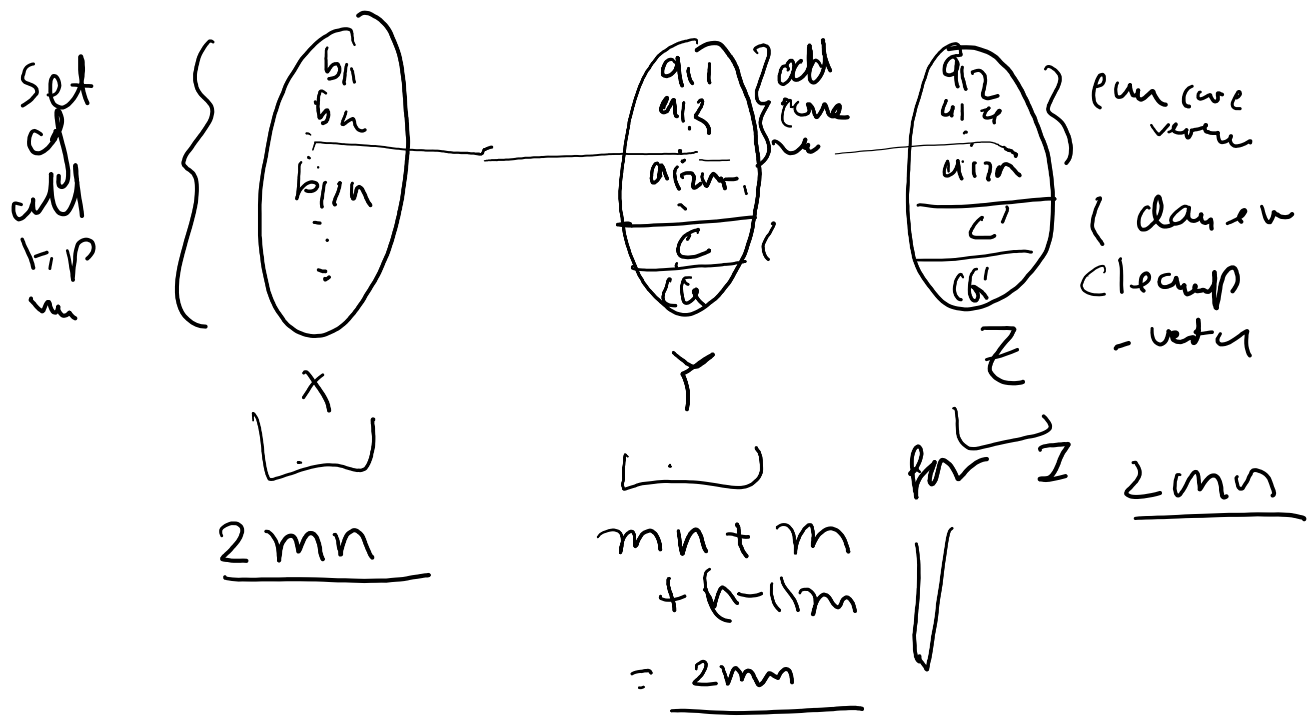
$x_i = T$, if variable gadget of x_i is oddly matched

$x_i = F$, if va.. gadget of x_i is evenly matched

∴ each clause vertices c_r, c_r' can be covered with the tip of some variable-gadget

⇓

assignment above will satisfy c_r in I



SUBSET-SUM

Given: 1) Multiset S of integers
2) target t.

Question: ∃ subset of S that sums up to t.

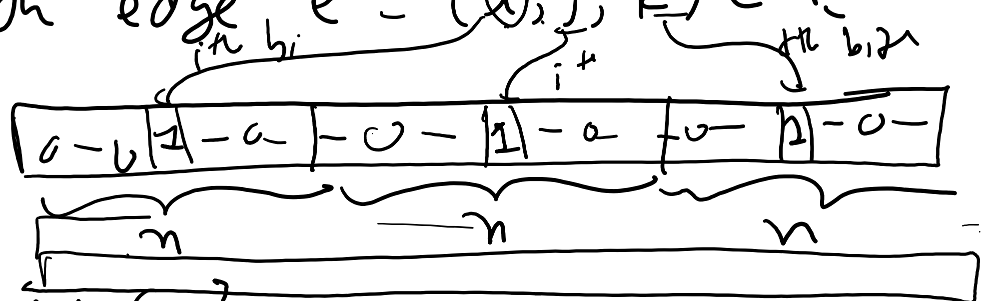
$O(nt)$

3DM $\leq_p$ SUBSETSUM

1) X, Y, Z
2) E
3) $\exists$ M that covers all vertices in $X \cup Y \cup Z$.

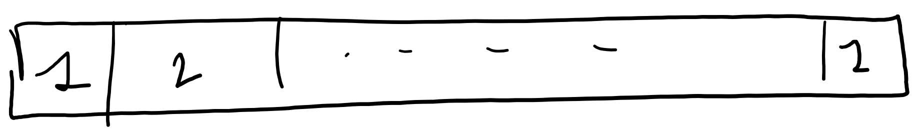
For each edge $e = (i, j, k) \in E$

$S_e =$



1) $\underline{S'} = \{ \bigcup_{e \in E} S_e \}$

2) $t = \sum_{j=1}^{3n} 2^j$



$\exists$ a matching M that covers all vertices of $X \cup Y \cup Z \iff \exists$ subset $S'' \subseteq S'$ that sums up to t.

$\Rightarrow$ let M be the match

$\underline{S''} = \bigcup_{e \in M} S_e$

$$\sum_{e \in M} s_e = t.$$

$\Leftarrow$

PARTITION - PROBLEM

Given: a multiset S of integers.

Question: Does there exist a partition of S into S_1 & S_2 s.t.

$$\sum_{s \in S_1} s = \sum_{s \in S_2} s = \frac{1}{2} \sum_{s \in S} s.$$

SUBSET-SUM $\leq_p$ PARTITION

1) S of integers $\longrightarrow$ 1) $S' = S \cup \{a\} \cup \{2t\}$

2) t

$a = \text{sum}(S)$

$\text{sum}(S') =$

$\text{sum}(S) + \text{sum}(a) + 2t$

$= 2\,\text{sum}(S) + 2t$

$= 2\,(\text{sum}(S) + t)$

$\exists S'' \subseteq S$ s.t. $\boxed{\sum_{s \in S''} s = \text{sum}(S) + t}$

Claim:→ $\exists$ a subset of S that sum upto t $\iff$ $\exists$ a partition of S''.

$\Rightarrow$ $Q \subseteq S$ s.t. $\sum_{s \in Q} s = t$.

Pick Q & $\{q\}$ from S'

$$\text{sum}(Q \cup q) = \text{sum}(Q) + \text{sum}(S)$$
$$= \boxed{\text{sum}(S) + t}$$

$\Leftarrow$ $T \subseteq S'$ that sums up to $\text{sum}(S) + t$

OBS:→ T contains at least one of $\{q\}$ or $\{2t\}$

T contains atmost one of $\{q\}$ or $\{2t\}$

CASE 1:→ if T contains $\{q\}$

$$\cancel{\text{sum}(S)} + t = \text{sum}(T) = \text{sum}(q) + \text{sum}(T \cap S)$$
$$= \cancel{\text{sum}(S)} + \text{sum}(T \cap S)$$

$$\Rightarrow \text{sum}(T \cap S) = t$$

CASE-2:→ if T contain $\{2t\}$

$$\mathrm{sum}(S) + t = \mathrm{sum}(T) = 2t + \mathrm{sum}(T \cap S)$$

$$\Rightarrow \underline{\mathrm{sum}(T \cap S) = \mathrm{sum}(S) - t}$$

$$\Rightarrow \mathrm{sum}(\underline{S \setminus (T \cap S)}) = t$$

KNAPSACK

PARTITION $\leq_p$ KNAPSACK
