More on NP completeness

Monday, April 18, 2022 2:33 PM

Recap :->

P = { problems that admit a polynomial time alg }.

NP = { Problems that admit a polynomial non-deterministic alg. }

= { Problems that admit a polynomial size certificate/proof & a polynomial time alg to verify the certificate for all YES inputs }

Reduction

$\underline{A} \leq_p \underline{B}$

$\hookrightarrow I \rightarrow I'$ (polynomial-time) A(I) is true $\Leftrightarrow$ B(I') is true

NP-complete-Problems :->
1) Problems in NP
2) Problems that are NP hard

Cook-Levin-Thm

SAT is NP-complete

Strategy :→ How to show a decision problem A is NP complete.

1) $A \in NP$

2) $SAT \leq_p A$.

$$SAT \leq_p 3SAT \leq_p CLIQUE$$

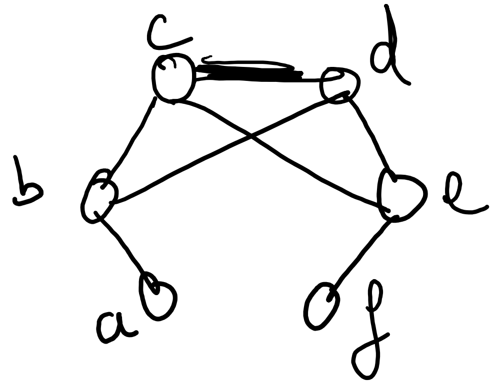
INDEPENDENT-SET PROBLEM

Given : 1) Graph $G = (V, E)$, $|V| = n$, $|E| = m$

2) $S \subseteq V$ is an independent set; if no two vertices in S share an edge.

3) Given k

Question :→ ∃ an independent set in G of size k ?



{a, e}, {b, f}, {a, f} = IS of size 2

{a, f, d}, {a, b, c} = IS of size 3

a c d f ≠ IS

Prove: IND-SET is in NP?

IND-SET is NP-complete

$SAT \leq_p CLIQUE \leq_p IND\text{-}SET$

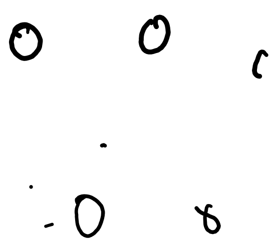
$I \downarrow (G, k) \longrightarrow I' \downarrow (H, k')$

Construct the graph $\overline{G}$

$(u,v) \in \overline{G} \iff (u,v) \notin G$

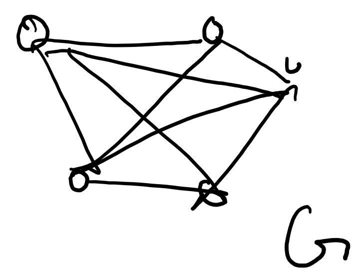
Claim: $\overline{G}$ has an independent set of size $k \iff G$ has a clique of size k.

$\Rightarrow$



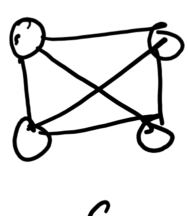
$\overline{G}$

$\Rightarrow$



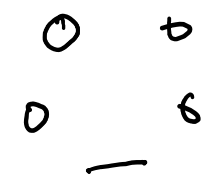
G

$\Leftarrow$



G

$\Rightarrow$



$\overline{G}$

VERTEX - COVER

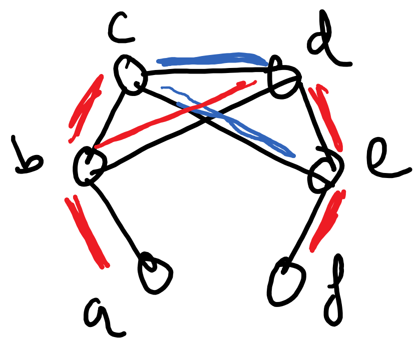
VERTEX - COVER

Given: 1) $G = (V, E)$ and $|V| = n$, $|E| = m$

2) $S \subseteq V$ is a vertex-cover iff $\forall$ edges $(u,v) \in G$, we have either $u \in S$ or $v \in S$.

3) Given a number k.

Question: $\exists$ vertex-cover of size k in G



$\{a, b, c, d, e, f\}$ = vertex-cover.

$\{b, c, d, e\}$ = vertex-cover

$\{b, c, e\}$ = vertex-cover

1) VERTEX-COVER is in NP

2) VERTEX-COVER is NP-hard.

IND-SET $\leq_p$ VERTEX-COVER

G, k $\qquad$ $G, n-k$

claim: G has an independ set of

size k $\Leftrightarrow$ G has a vertex cover of size $n-k$.

If S is a vertex cover in G

$\Updownarrow$

$V \setminus S$ is an independent set in G

$\Rightarrow$ Assume $V \setminus S$ is not an IS.

$\exists u, v \in V \setminus S$ s.t. $(u,v) \in E$

not covered by our vertex cover S ↯

$\Leftarrow$ Assume S is not a vertex-cover

$\Rightarrow \exists u, v \notin S$, s.t. $(u,v) \in E$

$\exists u, v \in V \setminus S$, $(u,v) \in E$

$\Downarrow$

$V \setminus S$ is not an independent set ↯

VERTEX-COVER is NP-COMPLETE

ILP (Integer Linear Programs)

$$Ax \leq b \quad x \in \mathbb{R}^n$$

x_i is integer for all $i \in [n]$

ILP is NP-hard

VERTEX-COVER $\leq_p$ ILP.

for every vertex $v \in G$, $y_v = 1$ if v is VC, 0 otherwise

$\forall (u,v) \in E \quad y_u + y_v \geq 1$

$\forall v \in V \quad y_v \in \{0,1\}$

$\sum_v y_v = k.$

} CONSTRAINTS

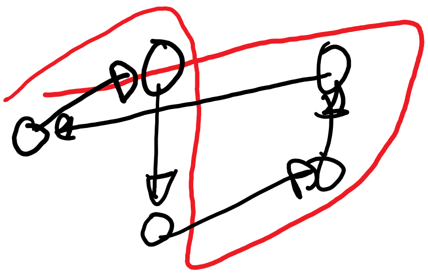
VERTEX-COVER $\leq_p$ ILP

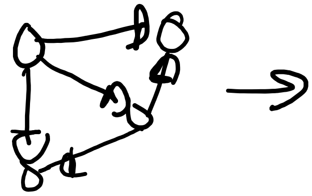
SAT $\leq_p$ 3SAT $\leq_p$ CLIQUE $\leq_p$ IS $\leq_p$ VC $\leq_p$ ILP

Hamiltonian cycles (Directed & undirected)

Given: Graph $G = (V, E)$ (Directed)

Question: $\exists$ exist a simple cycle in G that contains/visits all vertices





NO HAMILTON. CYCLES

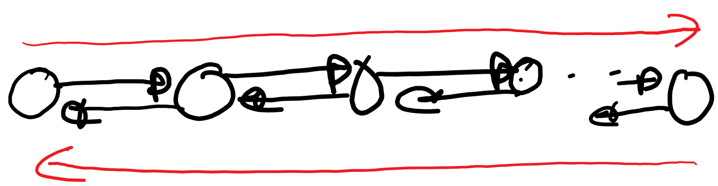
3SAT $\leq_p$ HAMILTONIAN CYCLE

variables - gadgets

clause - gadgets

Given instance I → $\{ x_1, x_2, \ldots, x_n$; $C_1, C_2, \ldots C_m \}$

x_i

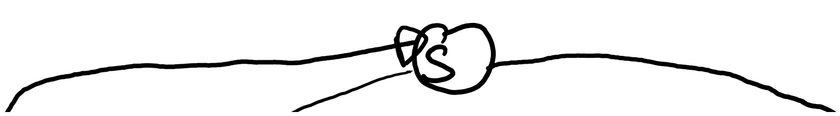


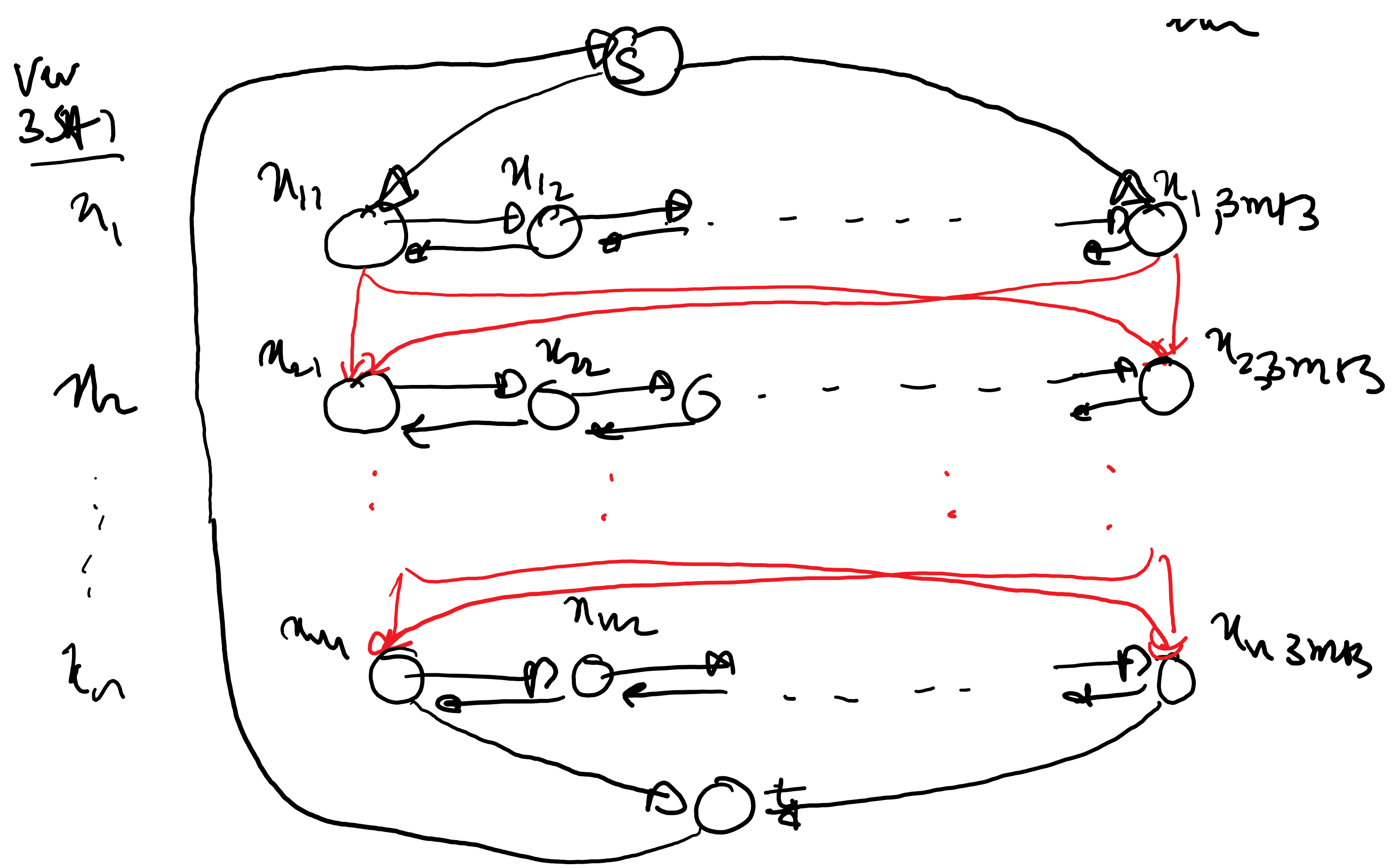
x_i

3m+3

↳ # clauses that are in

Vw





Hamiltonian - cycles

S → level 1 → level 2 → level n → t

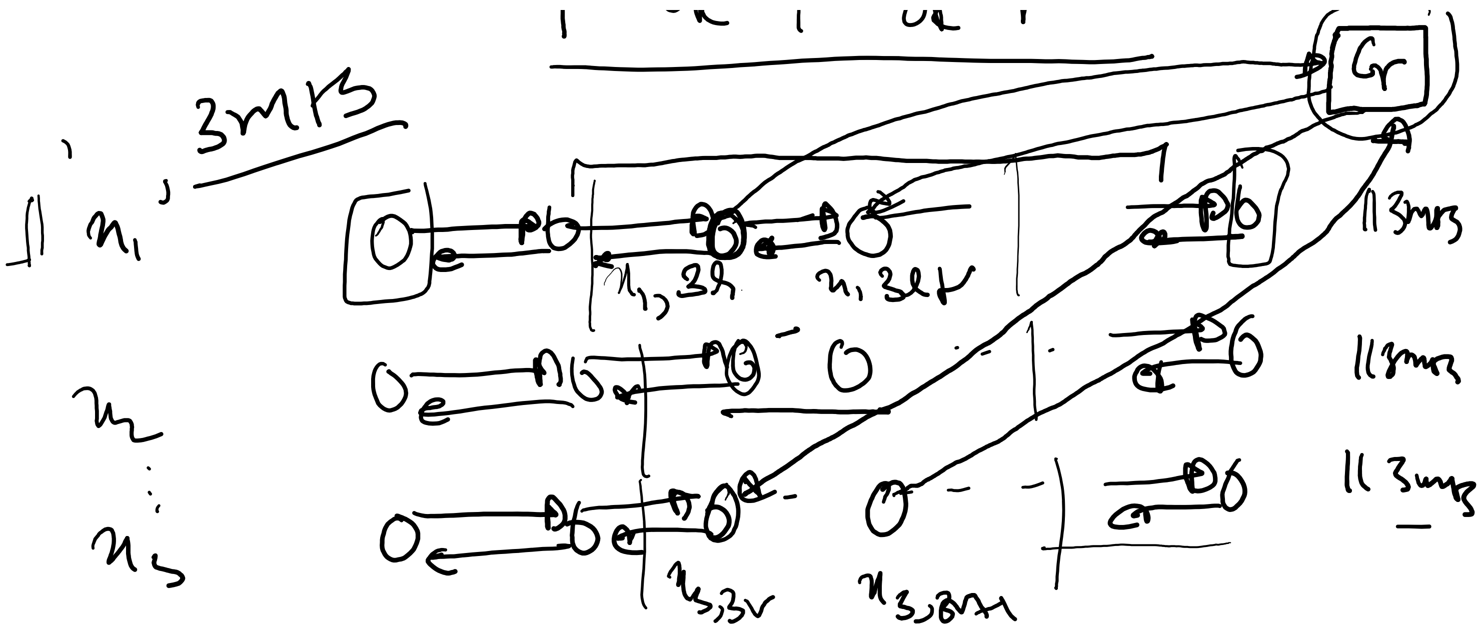
(traverse forward/backward) (' - '

$\#$ hamiltonian cycles $\geq 2^n$

Clause Gadgets:

$C_2 = (x_1 \vee x_2 \vee \neg x_3)$

T OR T OR F



each clause $C_\ell = (l_i \vee l_j \vee l_k)$

$\{x_i, \neg x_i\}$ $\{x_j, \neg x_j\}$ $\{x_k, \neg x_k\}$

draw edges

l_i: $x_{i,3r} \to C_\ell$ & $C_\ell \to x_{i,3r+1}$ if $l_i = x_i$

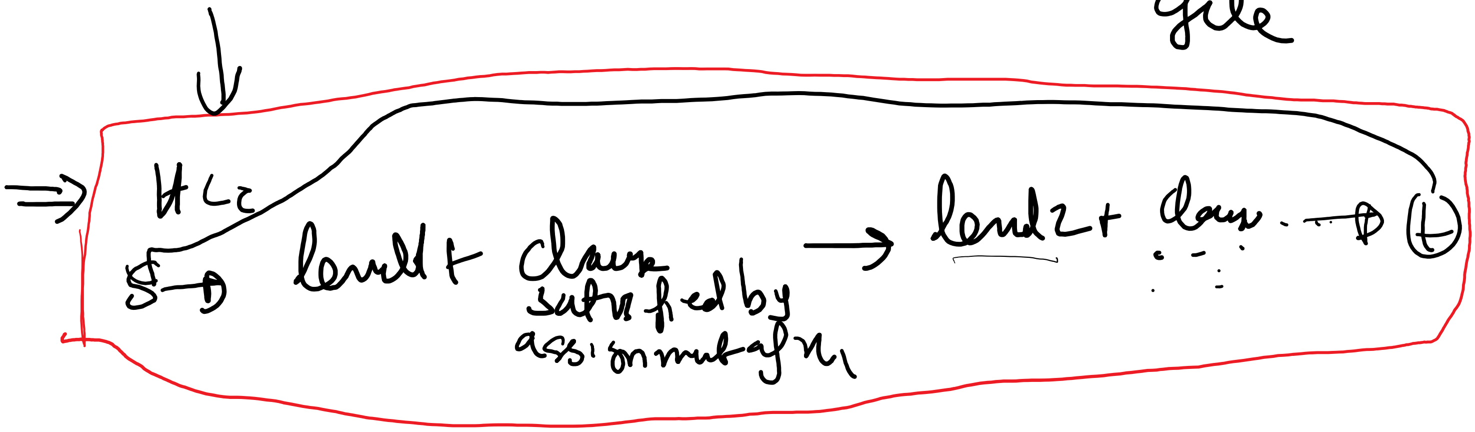
$x_{i,3r+1} \to C_r$ & $C_\ell \to x_{i,3\ell}$ if $l_i = \neg x_i$

l_j

l_k

Claim: I of 3SAT has a satisfying

assignment ⟺ G_2 has a hamiltonian cycle

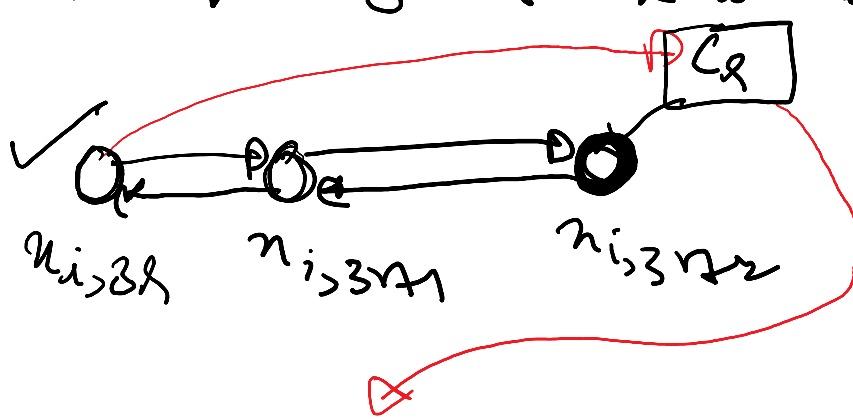


⟸ if ∃ Hamiltonian cycle H ⟹ a satisfying assignment.

Claim: For any clause $C_r = (l_i \vee l_j \vee l_k)$

1) if H visits C_r by edge from $x_{i,3r}$, then H leaves C_r using the edge to $x_{i,3r+1}$

2) if H visits C_r by edge from $x_{i,3r+1}$, then H leaves C_r using the edge to $x_{i,3r}$



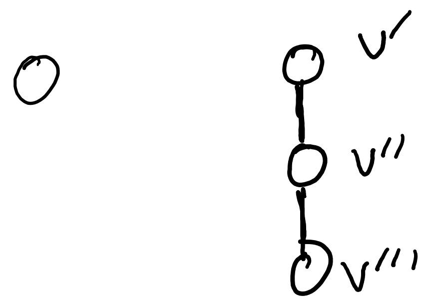
cannot maintain hamiltonicity.

I of 3SAT has a satisfying assignment iff G has a hamiltonian cycle.

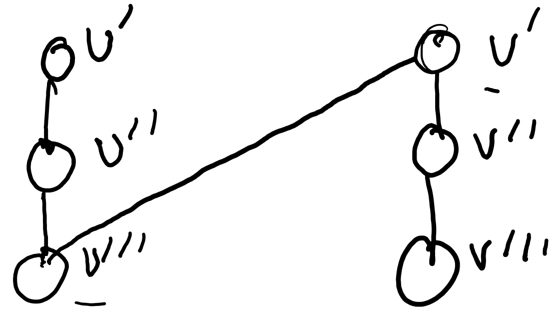
Undirected-Hamiltonian-cycle

Steps: →

1) For every vertex $v \in G$, replace v with 3 vertices v', v'', v'''



2) For every directed edge of the form $u \to v$



G'

Claim: G' has a hamiltonian cycle $\iff$ G has a hamiltonian cycle.

Recap